Geometrical characterization of non-Markovianity

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The interaction with an environment leads a quantum system to dissipate energy and lose its coherence. The process, however, needs not be monotonic and the system may temporarily recover some of the lost energy and/or information. This is the essence of a non-Markovian behavior, which can be characterized and quantified in different ways [1, 2]. One possibility is to look for temporary increases of the information, or coherence by the system. This is the essence of a non-Markovian behavior, which can temporarily recover some of the lost energy and/or information. The interaction with an environment leads a quantum system to dissipate energy and lose its coherence. The process, however, needs not be monotonic and the system may temporarily recover some of the lost energy and/or information. This is the essence of a non-Markovian behavior, which can be characterized and quantified in different ways [1, 2]. One possibility is to look for temporary increases of the information, or coherence by the system. This implies that the domain’s volume of the dynamical map decreases monotonically. On the contrary, we associate non-Markovianity of the dynamics to a growth of this domain’s volume. By properly formulating this intuition, we define a tool to quantify non-Markovianity, given by the sum of the (temporary) volume increases which occur during the time evolution.

In the case of a single qubit, this can be linked to the BLP measure [4], as the trace distance coincides with the Euclidean distance on the Bloch sphere and the pair of states that maximize the measure lie on the boundary of the convex subspace of physical states [11]. As a result, if the optimized trace distance decreases monotonically, so does the volume, which is, however, much easier to evaluate through the determinant of the dynamical map. We stress that ours is not another attempt at the quantification of the degree of non-Markovianity of a given dynamical process, but the proposal for another way to reveal physical effects (so far overlooked) of an evolution departing from Markovianity. Our proposal enjoys features of practicality and intuition of interpretation that are different from analogous, equally valid, quantifiers.

Systems with finite dimensional Hilbert spaces. Irrespective of the initial open system state, a reduced time evolution derived from the unitary dynamics of a larger system can always be described by a linear, Hermitian map [12], which is completely positive if there are no initial system-environment correlations [13]. In particular, a Markovian or memoryless behavior leads to master equations in the Lindblad form [14, 15], with the map obeying the semigroup composition law. A systematic construction of Gα’s is given in Refs. [16, 17] and leads to \(\{\hat{G}_a\}_{a=1}^{N^2-1} = \{\hat{u}_{jk}, \hat{v}_{jk}, \hat{w}_l\}/\sqrt{2}\) with

\[
\hat{u}_{jk} = |j\rangle\langle k| + |k\rangle\langle j|, \quad \hat{v}_{jk} = -i(|j\rangle\langle k| - |k\rangle\langle j|),
\]

\[
\hat{w}_l = \sqrt{\frac{2}{ll+1}} \sum_{j=1}^{l} (|j\rangle\langle j| - l(l+1)|l\rangle\langle l+1|) + \text{(terms with } l \text{ and } l+1 \text{ exchanged)}
\]

Here, \(\hat{\rho}\) is the quantum state of a N-level open system, which can be expressed through a generalized Bloch vector \(\mathbf{r}\), whose components are the expectation values of the traceless, Hermitian generators of SU(N), \(G_i(i = 1, \ldots, N^2 - 1)\), for which \(\text{Tr}[\hat{G}_i\hat{G}_j] = \delta_{ij}\). By including the identity \(\hat{G}_0 = \hat{1}/\sqrt{N}\), any state can be written as

\[
\hat{\rho} = \sum_{a=0}^{N^2-1} \text{Tr}[\hat{\rho}\hat{G}_a]\hat{G}_a = \sum_{a=0}^{N^2-1} r_a \hat{G}_a,
\]
where \(1 \leq j < k \leq N, 1 \leq l \leq N - 1\), and \(\{m\}_{m=1}^{N}\) is an orthonormal basis of the open system’s Hilbert space.

Writing the map (1) in this basis, one gets
\[
\tilde{r}_i = F(t)r_0, \quad \text{with} \quad F_{\beta\beta}(t) = \text{Tr}[\hat{G}_\alpha \phi_\beta | \hat{G}_{\beta\beta}]].
\]
(4)

As \(F_{\beta\beta}(t) = \delta_{\beta\beta}\), this is an affine transformation for the Bloch vector. Letting \(\phi_\beta(t) = F_{\beta\beta}\), we have
\[
F(t) = \begin{pmatrix}
1 & 0 \\
q_\beta & A_\beta \\
\end{pmatrix} \rightarrow r_i = A_i r_0 + q_\beta / \sqrt{N}.
\]
(5)

The real matrix \(A_\beta\) can be decomposed as \(A_\beta = O^\beta L_i O_i^T\), where \(O^\beta\)’s are orthogonal matrices and \(L_i\) is a positive semidefinite diagonal one. In what follows, we will indicate with \(|M|\) the determinant of a matrix \(M\). The findings above imply that \(|F_i| = |A_i| = |D_i|\). The action of \(F\) is given by a rotation (possibly composed with an inversion), a shrink of the Bloch vector, and a final rotation with possibly a translation. Its determinant gives the contraction factor of the volume of accessible states, given by the measure of the set of evolved Bloch vectors, with respect to its value at \(t = 0\).

The set of physical Bloch vectors for an \(N\)-level system is given by [18]
\[
B_N = \{ \text{r} \in \mathbb{R}^{N^2 - 1} : (-1)^j a_j (\text{r}) \geq 0 \ (j = 1, \ldots, N)\},
\]
where \(a_j\) are the coefficients of the characteristic polynomial \(\det(x I - \rho)\) with \(\rho = \frac{1}{N} \sum_{i=1}^{N^2 - 1} r_i G_i\). In spherical coordinates, the volume element of \(B_N\) is
\[
d^N V = \left| \frac{\partial (r_i)}{\partial (R_{\alpha \beta})} \right| dR d\phi_1 d\phi_2 \cdots d\phi_{N-1}.
\]
(6)

Any positive trace-preserving map described by Eq. (4) induces the change \(d^N V(t) = |A_i| d^N V(0)\). Therefore \(|A_i|\) describes the change in volume of the set of states accessible through the evolution of the reduced state. In particular, \(|A_i|\) decreases monotonically for any positive, linear, and trace-preserving map [19], and so it does for any element of a completely positive continuous one-parameter semigroup. Indeed, if \(\phi_t = \exp[t \hat{L}]\) with
\[
\hat{L} \rho = i[\rho, \hat{H}] + \sum_{\alpha, \beta} \gamma_{\alpha, \beta} \left( \hat{C}_\rho \hat{C}_\alpha \hat{C}_\beta^\dagger - \frac{1}{2} [\hat{C}_\rho \hat{C}_\alpha, \hat{C}_\beta] \right),
\]
(7)

\(\gamma_{\alpha, \beta} \geq 0\), and \(\hat{H} = \hat{H}^\dagger\), then \(|A_i|\) reduces to a constant.

Generators of the form in Eq. (7), but with explicitly time-dependent coefficients or jump operators lead to time-dependent Markovian processes. Indeed, although not being part of a dynamical semigroup, the corresponding dynamical map \(\phi_{t+\tau, t} = \exp[T_{\hat{D}} t^{t+\tau, \hat{L}} dt]\) (\(T_{\hat{D}}\) is the time-ordering operator) is divisible and can always be written as a composition of two CPT maps
\[
\tilde{\phi}_{t+\tau, 0} = \tilde{\phi}_{t+\tau, t} \tilde{\phi}_{t, 0} \quad (\forall \tau, t \geq 0).
\]
(8)

As a consequence, in this case too the determinant decreases monotonically.

These considerations lead us to define a way to quantify the non-Markovianity of a quantum evolution through the variation of the volume of accessible states
\[
\mathcal{N}_V = \frac{1}{V(0)} \int_{\mathcal{B}, V(t) > 0} \frac{d_\alpha V(t)}{d_\beta |F_\beta|} = \int_{\tilde{\phi}, |F_\beta| > 0} \tilde{\phi}_\beta |F_\beta|.
\]
(9)
of evolved states found from the contraction of $B_N$. This is an ellipsoid for the case of a qubit. As the number of states that any realistic experimental implementation can sample is finite, it would be a precious piece of information to know which are the best initial states to use in order to determine the set of physically accessible states at a given time $t$ and its volume. For this purpose, let us consider $N$ initial Bloch vectors, evolved up to time $t$ and arranged as the columns of a matrix $\mathbf{P}_t$. From Eq. (5), we find $\mathbf{P}_t = \mathbf{A}_t \mathbf{P}_0 + \mathbf{Q}_t$, where the columns of $\mathbf{Q}_t$ are given by $\mathbf{q}_t$’s (which is the evolved state for a maximally mixed initial condition). It then follows that $\left| \left[ \mathbf{P}_t - \mathbf{Q}_t \right] \right|^2 = \left| \left[ \mathbf{A}_t \mathbf{P}_0 \right] \right|^2$. Thus, if we choose as initial Bloch vectors the elements of any orthogonal basis in $\mathbb{R}^{N-1}$ plus the null vector corresponding to the maximally mixed state, then their time evolutions (arranged to form the matrix $\mathbf{P}_t - \mathbf{Q}_t$) gives the determinant of the map, from which the measure $\mathcal{N}_V$ easily follows. Therefore, the geometric measure of non-Markovianity in Eq. (9) can be revealed experimentally by performing a state tomo-graphy at different times for $N^2 - 1$ initial orthogonal states. This will be sufficient to evaluate the change in volume of the accessible states without the need for prior knowledge about the environment or the coupling. This is illustrated in Fig. 2(a) for a qubit: The vectors associated with the canonical basis of $\mathbb{R}^3$ are mapped onto the corresponding points on surface of the ellipsoid that comprises all the possible accessible states of the evolution. Incidentally, the problem of finding minimum volume covering ellipsoids has been studied extensively and efficient algorithms exist for the computation of such volume given $n$ points in $\mathbb{R}^N$ (a noticeable example being in Ref. [20]).

**Examples**

**Example 1:** Spontaneous emission into a leaky cavity. Consider a single two-level atom with transition frequency $\omega_0$ interacting with a vacuum electromagnetic field having a Lorentzian spectral density (mimicking a leaky cavity) [5,23]. Taking

$$J(\omega) = \frac{1}{2\pi \omega} \frac{\gamma_0 \lambda^2}{(\omega_0 - \Delta - \omega)^2 + \lambda^2},$$

where $\Delta$ is the detuning between the atomic and the cavity frequency, the atomic state at time $t$ reads

$$\hat{\rho}_A(t) = \left( \begin{array}{cc} |\Gamma(t)|^2 \rho_0^{++} & \Gamma(t) \rho_0^{-+} \\ \Gamma(t) \rho_0^{+-} & (1 - |\Gamma(t)|^2) \rho_0^{--} + \rho_0^{--} \end{array} \right),$$

with $\Omega_{\pm} = \Delta - i \lambda \pm \sqrt{(\Delta - i\lambda)^2 + 2 \gamma_0 \lambda}$ and

$$\Gamma(t) = e^{\frac{\gamma_0 t}{2 \Omega_{-} + e^{-\frac{\gamma_0 t}{2 \Omega_{-}}}}} e^{-\frac{\gamma_0 t}{2 \Omega_{+} - e^{-\frac{\gamma_0 t}{2 \Omega_{+}}}} - 2 \lambda \Gamma(t)}. \left(15\right)$$

The evolution of the Bloch vector is ruled by

$$\mathbf{A}_t = \left( \begin{array}{ccc} \text{Re} \Gamma(t) & \text{Im} \Gamma(t) & 0 \\ -\text{Im} \Gamma(t) & \text{Re} \Gamma(t) & 0 \\ 0 & 0 & |\Gamma(t)|^2 \end{array} \right),$$

whose determinant, $|\mathbf{A}_t| = |\Gamma(t)|^4$, is shown in Fig. 1 against the dimensionless time $\lambda t$. The corresponding non-Markovianity measure $\mathcal{N}_V$ is reported in Fig. 2(b), from which it is clear that a strongly non-Markovian behavior is found.
for a resonant coupling in the so-called good-cavity limit \( \gamma_0 \gg \lambda \). A similar result is obtained with the RHP measure, [3] which is given by the integral of \((1/2)\text{Re} \ln \Gamma(t)\). This is in agreement with BLP as well, which turns out to depend on \(|\Gamma(t)|\) [5].

**Example 2: Pure dephasing.** Let us consider a qubit undergoing a purely dephasing dynamics, expressed in terms of a decoherence factor \( \nu(t) \)

\[
\hat{\varphi}^{(d)}(t) = \left( \begin{array}{cc} \rho^{++} & \nu(t)\rho^{+-} \\ \nu(t)\rho^{-+} & \rho^{--} \end{array} \right). 
\]

In this case, the BLP and RHP measures coincide [24,25].

Following Ref. [26], the dynamics induced by the chosen symplectic transformations to describe the evolution [22]. and thus use the formalism of covariance matrices and allows us to restrict our analysis to input Gaussian states for which \( N_{\text{m}} \) mode rotations in phase space.

**Example 3: Harmonic oscillators.** We consider a single quantum harmonic oscillator that interacts with a reservoir of \( N_{\text{m}} \) modes as

\[
\hat{H} = \omega_0 (\hat{q}_0^2 + \hat{\rho}_0^2) + \omega \sum_{j=1}^{N_{\text{m}}} (\hat{q}_j^2 + \hat{\rho}_j^2) + \kappa q_0 \sum_{j=1}^{N_{\text{m}}} \hat{q}_j. 
\]

When expressed in terms of bosonic annihilation and creation operators, the interaction term in Eq. (19) includes both rotating and counter-rotating terms. The latter do not preserve energy and induce squeezing in the system, in turn resulting in an effective growth of the determinant of the map above its initial value. In what follows, we will work assuming that the total energy of the system is fixed, and thus invoke a rotating-wave approximation that allows us to neglect counter-rotating terms and reduce the system-environment coupling to the generator of beamsplitterlike transformations \( \kappa \hat{q}_0 \sum_{j=1}^{N_{\text{m}}} \hat{q}_j \rightarrow (\kappa/2) \hat{q}_0 \sum_{j=1}^{N_{\text{m}}} \hat{q}_j + \text{H.c.} \). The quadratic nature of this model allows us to restrict our analysis to input Gaussian states and thus use the formalism of covariance matrices and symplectic transformations to describe the evolution [22].

Following Ref. [26], the dynamics induced by the chosen model transforms the input covariance matrix \( \sigma_0 \) of the \( N_{\text{m}} + 1 \)-mode system as \( \sigma_t = T \sigma_0 T^T \) with the symplectic transformation \( T = T_{\text{coll}} T_{\text{coll}}^{\dagger} T_{\text{coll}}^{\dagger} \) composed of a series of pairwise beamsplitters \( T_{\text{coll}} \) and the tensor product of single-mode rotations in phase space \( T_{\text{coll}} \). We refer to Ref. [26] for the form of such transformations.

Here, it is enough to mention that, by choosing a factorized initial state for the system and reservoir, the evolution of the central oscillator can be found by tracing over the degrees of freedom of the \( N_{\text{m}} \) environmental modes and can be cast into the form of Eq. (10) with

\[
X_t = \begin{pmatrix} \sqrt{T_{11}^2 + T_{12}^2} & -T_{11} T_{31} + T_{31} T_{32} & T_{11} T_{32} - T_{31} T_{32} & 0 \\ T_{12}^2 & \sqrt{T_{11}^2 + T_{12}^2} & T_{12} T_{31} + T_{31} T_{32} & 0 \\ T_{31} T_{32} - T_{11} T_{32} & -T_{31} T_{32} + T_{11} T_{32} & \sqrt{T_{31}^2 + T_{32}^2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} 
\]

and \( T_{ij} \)'s the entries of the symplectic matrix associated with \( T \). We finally have \( |X_t| = |T_{12} T_{21} - T_{11} T_{22}|^2 \), whose behavior within a fixed time interval of the evolution is shown in Fig. 3, evidencing non-Markovianity due to the backflow of excitations to the central mode.

**Conclusions.** We have proposed a geometrically motivated quantifier of non-Markovianity \( N_{\nu} \) explicitly linked to the variations of the volume of the physical states dynamically accessible by a given open system. From an information theoretical perspective, such a measure is linked to the loss or regain of classical information over the evolving system. We have shown how \( N_{\nu} \) can be estimated through only a polynomial number of measures, and that it enjoys a straightforward extension to the Gaussian continuous-variable scenario. Finally, we have illustrated our proposal through three examples that emphasize its appealing aspects of practicality and intuitive nature.

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