Coherently Opening a High-\(Q\) Cavity

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We propose a general framework to effectively “open” a high-\(Q\) resonator, that is, to release the quantum state initially prepared in it in the form of a traveling electromagnetic wave. This is achieved by employing a mediating mode that scatters coherently the radiation from the resonator into a one-dimensional continuum of modes such as a waveguide. The same mechanism may be used to “feed” a desired quantum field to an initially empty cavity. Switching between an open and “closed” resonator may then be obtained by controlling either the detuning of the scatterer or the amount of time it spends in the resonator. First, we introduce the model in its general form, identifying (i) the traveling mode that optimally retains the full quantum information of the resonator field and (ii) a suitable figure of merit that we study analytically in terms of the system parameters. Then, we discuss two feasible implementations based on ensembles of two-level atoms interacting with cavity fields. In addition, we discuss how to integrate traditional cavity QED in our proposal using three-level atoms.

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Introduction.—The past two decades have witnessed the blooming of cavity QED, through vast advances in the development of high-\(Q\) optical and microwave cavities, and in the ability to coherently control individual quantum emitters interacting with confined radiation [1–3]. Cavity QED has long been the paradigmatic setup to investigate models of interaction between light and matter at the single-photon level, and led both to investigations into the fundamental properties of quantized radiation [4] and to the development of some of the most sophisticated quantum control techniques available to date [5]. Recently, analogous models have been implemented in a variety of experimental platforms such as circuit QED, trapped ions, and Bose-Einstein condensates (BECs) [6–8].

Promising as this progress may be, the step from proof of principle demonstrations to operational quantum technology inspired by the cavity QED paradigm is prevented at this stage by a fundamental difficulty: on one hand, such an endeavor would request one to operate light matter interactions in the strong coupling regime, where the coupling strength is at least comparable to the cavity decay rate; on the other hand, it would be highly desirable to extract the state of the cavity field on reasonably short time scales. These two requirements, implying respectively high and low \(Q\) factors, are inherently contradictory. In addition, in high-\(Q\) cavities, the photon lifetime can be maximized only by reducing the transmittivity to a minimum, typically to values comparable to the cavity losses. Thus, the physically accessible field that naturally leaks out from the cavity does not faithfully retain the quantum properties of the intracavity field, posing a major problem for the exploitation of cavity QED-like architectures in scalable quantum information processing and quantum networks [9].

In light of the above, it would be extremely desirable to control in time the \(Q\) factor of a cavity, possibly switching between a cavity in the strong coupling regime and an “open” one in a coherent fashion. To this end, theoretical and experimental advances have been achieved in photonic crystal cavities [10], and some degree of control at the quantum level has been very recently demonstrated in superconducting resonators [11] and in optical cavities [12]. Let us stress that these efforts differ from usual studies on qubit networks [26], despite the fact that the latter often require to release and catch photons between cavities. In fact, the former aim at converting the field confined into a resonator—a continuous-variable system—to a traveling field, whereas the latter try to exchange information between confined two-dimensional systems (e.g., two-level atoms trapped in cavities with a fixed \(Q\) factor).

In this Letter, we propose a general framework to achieve such coherent control of a resonator \(Q\) factor, by introducing a mediating bosonic mode that scatters coherently the cavity radiation into an experimentally accessible, one-dimensional continuum of modes (waveguide for brevity). We quantify the performance of our scheme in terms of a few effective model parameters, and discuss some possible implementations based on the cavity-QED architecture. The complementary process of “feeding” an initially empty cavity through the waveguide is also studied and shown to yield the same performance [13].

The basic model.—The system under investigation is sketched in Fig. 1. We consider a high-\(Q\) resonator (cavity for brevity) in which a desired quantum field has been
prepared in advance. In order to switch the quality factor, the cavity is brought to interact with a bosonic scatterer which, in turn, radiates into an accessible waveguide. In this way the initial quantum state of the cavity can be coherently transferred to a traveling mode of light, thus effectively “opening” the cavity. To gain advantage from such a scheme, one has to be able to control the coupling between scatterer and cavity on short time scales: this may be obtained, e.g., by applying a detuning to the scatterer [14], or by controlling how much time it spends in the resonator. If these requirements are met, one can switch between a “closed” and an open cavity on demand. We describe the (single-mode) cavity field via the annihilation operator \( \hat{a} \) with \([\hat{a}, \hat{a}^\dagger] = 1\) and the bosonic scatterer by a second annihilation operator \( \hat{b} \). Their interaction Hamiltonian, in a frame rotating at the cavity frequency \( \omega_c \), is assumed of the form (\( \hbar = 1 \))

\[
H = g (\hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a}),
\]

where \( g \) is the coupling strength. The interaction between the scatterer and the continuum of waveguide modes, characterized by an emission rate \( \gamma \), is conveniently dealt with in the framework of input-output theory [15–17]. In addition, we take into account cavity losses at rate \( \kappa \) and the spontaneous emission of the scatterer into inaccessible modes—such as field modes that do not couple to the waveguide, or other internal degrees of freedom of the scatterer outside our control—at a rate \( \gamma_{\text{ext}} \) [see Fig. 1]. With standard assumptions [15], one can derive the Heisenberg-Langevin equations

\[
\dot{\hat{a}} = -i g \hat{b} - \frac{\kappa}{2} \hat{a} + \sqrt{\kappa} \hat{a}_{\text{in}},
\]

\[
\dot{\hat{b}} = -i g \hat{a} - \frac{\gamma + \gamma_{\text{ext}}}{2} \hat{b} + \sqrt{\gamma} \hat{b}_{\text{in}} + \sqrt{\gamma_{\text{ext}}} \hat{b}_{\text{ext, in}},
\]

where all operators are time dependent (the time variable \( t \) will be explicitly indicated only when its omission might be misleading). In particular, \( \hat{b}_{\text{in}} \) is associated with the waveguide modes under our control, while \( \hat{a}_{\text{in}}, \hat{b}_{\text{ext, in}} \) with inaccessible environments providing losses. All input modes are characterized by two-time commutators of the form \([\hat{a}_{\text{in}}(t), \hat{a}_{\text{in}}^\dagger(t')] = \delta(t-t')\), with analogous expressions for \( \hat{b}_{\text{in}}, \hat{b}_{\text{ext, in}} \). As our focus shall be the system’s output into the waveguide, it becomes convenient to work with the output operator \( \hat{b}_{\text{out}}^\dagger = \sqrt{\gamma} \hat{b} - \hat{b}_{\text{in}} \). This is characterized by the same commutation rules as \( \hat{b}_{\text{in}} \) [15], and conveniently describes the waveguide modes affected by the emission of the system (\( \hat{a}_{\text{out}}, \hat{b}_{\text{ext, out}} \) are defined by analogous equations but are not associated with detectable modes). We can thus rephrase Eqs. (2) and (3) in terms of these output fields. For convenience, let us define \( \hat{v} = (\hat{a}, \hat{b})^T \), and \( \hat{v}_{\text{out}} \equiv (\sqrt{\kappa} \hat{a}_{\text{out}}, \sqrt{\gamma} \hat{b}_{\text{out}} + \sqrt{\gamma_{\text{ext}}} \hat{b}_{\text{ext, out}})^T \). The equations of motion then read [15]

\[
\dot{\hat{v}} = \textbf{M} \hat{v} - \hat{v}_{\text{out}},
\]

\[
\textbf{M} \equiv \begin{pmatrix} \frac{\kappa}{2} & -ig \\ -ig & \frac{\gamma + \gamma_{\text{ext}}}{2} \end{pmatrix}.
\]

**Opening the cavity.**—Having fixed the notation, we can tackle the problem of opening the cavity as follows. At time \( t = 0 \), we assume that the cavity field has been prepared in a quantum state of interest, while all other relevant modes are in the vacuum. Equation (4) can be formally integrated between times \( t_0 \) and \( t_1 \) as \( \hat{v}(t_1) = e^{\textbf{M}(t_1-t_0)} \hat{v}(t_0) - e^{\textbf{M}t_0} \int_{t_0}^{t_1} dt' e^{-\textbf{M}t'} \hat{v}_{\text{out}}(t') \). Crucially, this expression is valid also when \( t_0 > t_1 \). Taking \( t_1 = 0 \) and \( t_0 \to \infty \), and using the stability condition \( \lim_{\tau \to \infty} e^{-\textbf{M} \tau} = 0 \), one has

\[
\hat{v}(0) = \int_0^\infty dt e^{-\textbf{M}t} \hat{v}_{\text{out}}(t),
\]

which relates the system operators at time \( t = 0 \) to specific combinations of the output fields in the time interval \([0, \infty]\). Expanding the first component of this vectorial identity as \( \hat{a}(0) = \sqrt{\gamma} \int_0^\infty dt (e^{-\textbf{M}t})_{1,2} \hat{b}_{\text{out}}(t) + [\text{inaccessible modes}] \), one can recast it in terms of canonical bosonic operators as

\[
\hat{a}(0) = \sqrt{F} \hat{f}_{\text{out}} - \sqrt{1 - F} \hat{h}_{\text{ext}},
\]

where \( \hat{f}_{\text{out}} \equiv \int_0^\infty dt u(t) \hat{b}_{\text{out}}(t) \) is a canonical bosonic mode of temporal profile \( u(t) \equiv (e^{-\textbf{M}t})_{1,2}/([\int_0^\infty dt (e^{-\textbf{M}t})_{1,2}]^{1/2}) \), which propagates away from the system along the waveguide, while \( \hat{h}_{\text{ext}} \) is a canonical bosonic mode representing the portion of the field that has been dissipated into the inaccessible modes \( a_{\text{out}}, b_{\text{ext, out}} \) (we do not concern ourselves with the specific form of \( \hat{h}_{\text{ext}} \), its sign being chosen for later convenience). The parameter \( F \), verifying \( 0 \leq F \leq 1 \) by construction, is given by
We note the mapping between Eq. (7) and a beam splitter [18] of transmittivity $F$, where $F_{\text{out}}$ and the inaccessible mode $\hat{a}_{\text{vac}}$ are mixed. Since all field modes except $\hat{a}(0)$ were initially in the vacuum, and the global evolution conserves the total excitation number [15], it follows that at the other output of this abstract beam splitter one must find the vacuum. That is, the relation $\sqrt{1-F}F_{\text{out}} + \sqrt{F}F_{\text{ext}} = \hat{a}_{\text{vac}}$ must hold, with $\hat{a}_{\text{vac}}$ a canonical bosonic mode in the vacuum state. Therefore, by inverting these relationships, one is finally able to express
\[
F_{\text{out}} = \sqrt{F} \hat{a}(0) + \sqrt{1-F} \hat{a}_{\text{vac}}.
\]

The explicit identification of the mode $F_{\text{out}}$ is a crucial result since it provides by construction the traveling mode that best retains the quantum information of the cavity field $\hat{a}(0)$. Clearly, the larger $F$ is, the closer the output field $F_{\text{out}}$ is to the initial cavity field. In addition, the Schrödinger picture interpretation of Eq. (9) is straightforward: suppose the cavity is prepared in the state $\rho_0$ at time $t = 0$; the mode $F_{\text{out}}$ is then found in the state $\rho_{\text{out}} = e^{-(1-F)\lambda \rho_0}$, where $\lambda = 1 (2\hat{a}\hat{a}^\dagger - \hat{a}^\dagger \hat{a} - \rho \hat{a}^\dagger \hat{a})$. As an instructive example, in [13] we apply these ideas to study the extraction of squeezed light from a cavity. To summarize, we mapped the channel. These are defined by a single parameter $\kappa$, the cooperativity parameter, which for convenience we write as a high figure of merit of our scheme. Defining $\gamma_{\text{tot}} \equiv \gamma + \gamma_{\text{ext}}$ one finds [13]
\[
F = \frac{1 - \gamma_{\text{tot}}}{1 + \frac{\kappa}{\gamma_{\text{tot}}} + \frac{\kappa \gamma}{4\gamma_{\text{tot}}} + \frac{\gamma^2}{4\gamma_{\text{tot}}}}.
\]

Notice that $F$ is monotonically decreasing in $\kappa$, and for an ideally closed cavity ($\kappa = 0$) the $Q$ switch approaches a perfect extraction of the cavity field, provided $\gamma_{\text{ext}} \ll \gamma_{\text{tot}}$. In other words, “the more a cavity is closed, the better it can be opened.” More in detail, Eq. (10) illustrates the constraints that the system has to satisfy in order to obtain a high figure of merit $F \sim 1$. The conditions $\gamma_{\text{ext}} \ll \gamma, \kappa \ll \gamma$, and $\kappa \ll g$ trivially state that the decay rates into inaccessible modes should be small, as compared to the strength of the desired interactions $g, \gamma$. A more specific condition can be identified, which for convenience we write as $4g^2/\kappa \gamma_{\text{tot}} \gg 1$. Drawing an analogy with standard cavity QED, this may be interpreted as the requirement of a large cooperativity parameter for the cavity-scatterer system: despite the constructive role of the emission rate $\gamma$, we are still requiring the system to be in a form of strong coupling regime. Quite remarkably, we find that a similar performance may be obtained if the bosonic mediator is replaced by a two level system whose excited state is only virtually populated [13].

Two-mode implementations.—Several implementations of our scheme can be envisaged, depending on the specific physical system that constitutes the high-$Q$ resonator. We speculate here on two possible implementations based on the interaction between an ensemble of atoms and two field modes of different lifetime, as sketched in Fig. 2. As before, $\hat{a}$ indicates the high-$Q$ cavity mode, while $\hat{a}_2$ represents a second field mode of the same frequency. We assume that the decay of the latter is associated with emission into a waveguide at rate $\eta$, plus some optical loss at rate $\eta_{\text{ext}}$. As shown in Fig. 2, the two fields may belong to different cavities, or they could be two distinct modes of the same cavity, e.g., with different polarization (in this case, the mirror transmittivity has to be different for the two modes to allow $\kappa \neq \eta$).

**Setting 1:** We consider an ensemble of $n$ two-level atoms that are brought to resonance with modes $\hat{a}$ and $\hat{a}_2$ when the high-$Q$ mode needs to be extracted. When instead a closed cavity is required, one can either apply a large detuning to the atoms or remove them altogether. We take the atoms to be initially in the ground state and identically coupled to the cavity fields, such that the interaction picture Hamiltonian reads
\[
H_1 = \sum_{k=1}^{n} \left[ \lambda (\hat{a}_k^\dagger \hat{a}_k - \hat{a}_k^\dagger \hat{a}_k) + \lambda' (\hat{a}_k^\dagger \hat{a}_k + \hat{a}_k^\dagger \hat{a}_k) \right],
\]
where $\hat{a}_k^\dagger = (\hat{a}_k^\dagger)^\dagger = \left| e_k \right\rangle \langle g_k |$ and $\left| e_k \right\rangle, \left| g_k \right\rangle$ are the excited and ground states of the $k$th atom, whereas $\lambda$ ($\lambda'$) denotes the coupling between the atoms and the $\hat{a}$ ($\hat{a}_2$) field. We assume here the Holstein-Primakoff regime, where $n$ is large enough, and the majority of atoms remain in the ground state during the interaction (in particular, this is guaranteed when the initial cavity excitation $\langle \hat{a}^\dagger \hat{a} \rangle$ is smaller than the total number of atoms $n$), such that a collective (approximately) bosonic operator $\hat{c} = (1/\sqrt{n}) \sum_{k=1}^{n} \hat{a}_k$ can be introduced (with $[\hat{c}, \hat{c}^\dagger] = 1$, see [20]). Denoting with $\Gamma$ the atomic decay rate into inaccessible modes, one has

![FIG. 2 (color online). An atomic ensemble interacts with two field modes of different lifetime. Left: The fields $\hat{a}, \hat{a}_2$ belong to two different cavities. The sketch is inspired by fiber-cavity setups [19], where an optical fiber can provide both the waveguide and the transmissive mirror of the second cavity. Right: The two modes belong to the same cavity, and are distinguished by some relevant degree of freedom such as polarization. While mode $\hat{a}$ is long lived, mode $\hat{a}_2$ is significantly transmitted through one of the cavity mirrors.](133605-3)
that the evolution of the three bosonic operators \( \hat{\mathbf{v}} \equiv (\hat{a}, \hat{a}_2, \hat{c}) \) is now given by
\[
\hat{\mathbf{v}} = \hat{\mathbf{M}}' \hat{\mathbf{v}}' - \hat{\mathbf{v}}_{\text{out}} \text{ where } \hat{\mathbf{v}}_{\text{out}} \equiv \left( \sqrt{\kappa} \hat{a}_{\text{out}}, \sqrt{\eta_{\text{ext}}} \hat{a}_2, \hat{c}_{\text{out}} \right),
\]
\[\hat{\mathbf{M}}' = \begin{pmatrix}
\frac{\kappa}{2} & 0 & -i\kappa \sqrt{n} \\
0 & \frac{\eta_{\text{ext}}}{2} & -i\kappa \sqrt{n} \\
-i\kappa \sqrt{n} & -i\kappa \sqrt{n} & \frac{-\kappa}{2}
\end{pmatrix}, \quad (12)
\]
and \( \eta_{\text{tot}} \equiv \eta + \eta_{\text{ext}} \). Our model can then be obtained by assuming the second cavity to be in the Purcell regime (or low-\( Q \) regime) [21], namely, \( \eta_{\text{tot}} \gg \lambda / \sqrt{n} \), which allows the mode \( \hat{a}_2 \) to be adiabatically eliminated. After this operation, upon identifying \( \hat{b} \equiv \hat{c} \) and \( \hat{b}_{\text{out}} \equiv \hat{a}_{2,\text{out}}, \) one can finally recover Eqs. (4) and (5) with \( g = \sqrt{\eta} \lambda, \gamma = 4n\lambda^2 \eta / \eta_{\text{tot}}, \gamma_{\text{ext}} = \Gamma + 4n\lambda^2 \eta_{\text{ext}} / \eta_{\text{tot}} \). Thus, the atomic ensemble takes the role of the bosonic scatterer, while the second mode provides a means to collimate the atomic radiation into the modes of interest. Taking a step further, we find it worthwhile to study the full model described by Eq. (12). Carrying out an analysis analogous to that leading to Eq. (9), one arrives again at the conclusion that the process of opening the cavity can be mapped to a beam splitter, with transmittivity \( \mathcal{F}' \) given by
\[\mathcal{F}' = T \eta_{\text{tot}} \int_0^\infty dt |(e^{-\mathcal{M}t})_{1,2}|^2, \quad (13)\]
where \( T \equiv \eta / \eta_{\text{tot}} \) is the waveguide coupling efficiency. This provides a more refined description of the process, valid beyond the Purcell regime. The analytical expression for \( \mathcal{F}' \) is given in [13] and proves to be rather involved. Still it retains the relevant feature of approaching \( T \) for vanishing \( \Gamma \) and \( \kappa \).

In Fig. 3 we report a case study inspired by the BEC-cavity system demonstrated in Ref. [22]. We assume to add an auxiliary cavity \( \hat{a}_2 \) to the setup, while leaving all other parameters unchanged. Due to the properties of BECs [8], every atom in the ensemble experiences an identical coupling to the cavity field, so that Eq. (11) directly applies. In the left panel we show the behavior of \( \mathcal{F}' / T \) as a function of the parameters \( \lambda', \eta_{\text{tot}} \) characterizing the auxiliary cavity. The right panel shows \( \mathcal{F}' / T \) as a function of the number of atoms \( n \) and the cavity decay rate \( \kappa \), having fixed \( \lambda', \eta_{\text{tot}} \) close to their optimal values. Again, we can clearly see that \( \mathcal{F}' \) monotonically increases as \( \kappa \) decreases. Because of its trivial effect on the protocol performance, the coupling efficiency \( T \) has been left implicit.

Setting 2: Hamiltonian (1) can be engineered by coupling the \( n \) atoms off resonantly to both fields \( \hat{a} \) and \( \hat{a}_2 \) (see, e.g., Ref. [23]). At variance with the previous case no bosonization of the atoms is required; hence, the scheme may be applicable also when the number of atoms \( n \) is small. We consider Eq. (11) with \( \lambda' = \lambda \), and add a detuning \( \Delta \) to all atoms, resulting in a Hamiltonian \( H_2 = H_1 + \Delta \sum_{k=1}^n |e_k \rangle \langle e_k | \). In the large detuning limit
\[\Delta \gg \lambda / \sqrt{n(\hat{a} \hat{a}')} \]

one can adiabatically eliminate the atomic excited states and derive an effective Hamiltonian for the ground state subspace [24,25]:
\[H_{\text{eff}} = -n(\lambda^2 / \Delta) |\hat{a}^\dagger \hat{a} + \hat{a}^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a} + \hat{a}_2^\dagger \hat{a}_2| \].

Identifying \( \hat{b} \equiv \hat{a}_2 \), this provides the desired interaction Hamiltonian (1), with \( g = -n\lambda^2 / \Delta \) (plus a global frequency shift which can be ignored).

Thus, in this setting the bosonic scatterer is provided by the second field, with \( \gamma \equiv \eta, \gamma_{\text{ext}} \equiv \eta_{\text{ext}} \) and one can control the strength of the coupling via the atomic detuning. The detrimental effects associated to atomic spontaneous emission can be estimated via the techniques of Ref. [25]. The adiabatic elimination of the atomic excited states results in an effective decay rate \( \kappa' \equiv n \Gamma / \lambda / \Delta \) affecting the superposition of fields \( \hat{a} + \hat{b} \). This requires the modification of Eq. (5) as per
\[M \rightarrow M'' = \begin{pmatrix}
\frac{\kappa + \kappa'}{2} & -ig + \kappa' \frac{i}{2} \\
-ig + \kappa' \frac{i}{2} & \eta_{\text{ext}} \kappa' \frac{i}{2}
\end{pmatrix}. \quad (14)
\]
As before, we can identify a figure of merit \( \mathcal{F}'' = T \gamma_{\text{tot}} \int_0^\infty dt |(e^{-\mathcal{M}t})_{1,2}|^2 \) (see [13] for its full expression). Studying this quantity with the same parameters reported in Fig. 3, and fixing \( \Delta = 5\lambda / \sqrt{10n} \), which guarantees the consistency of our approximations for cavity states with \( \langle \hat{a}^\dagger \hat{a} \rangle \lesssim 10 \), we find \( \mathcal{F}'' \approx 0.887T \) for \( n = 1000 \) and \( \eta = 4/2 \).

Integrating standard cavity QED.—A natural question to ask is whether the standard protocols of cavity QED are still applicable in our two-mode scheme. Furthermore, it would be convenient if the same atoms employed for the \( Q \)-factor control could be used for this purpose. In [13] we show that this is indeed possible, by addressing two internal transitions of the atoms such that each mode is coupled to a different transition. By applying appropriate Stark shifts to the atoms, one is then able to control whether the atoms interact with mode \( \hat{a} \) only, realizing cavity QED in the
strong coupling regime, or with both fields $\hat{a}$, $\hat{a}_2$, as required for our $Q$-switching proposal.

Conclusions and outlook.—We have proposed a general scheme in which a mediator allows us to switch coherently from a closed to an open cavity, so that the advantages of both regimes may be combined in a single setup. After having identified the accessible output mode that best represents the initially prepared cavity field, we have fully characterized the effective transmittivity parameter which encodes the quality of the process. As clarified in [13], the same figure of merit is obtained for the complementary process of feeding an initially empty cavity. Let us also emphasize that our scheme is applicable to a single qubit mediator whose excited level is only virtually populated, which allows for an effective bosonic description [13]. By considering a cavity-QED implementation we have shown that state of the art experimental parameters should be compatible with a demonstration of our scheme.

Our work may represent a contribution towards the achievement of ambitious goals such as the direct access to nonclassical cavity field states, the realization of cavity-based quantum memories and continuous-variable quantum networks. Goals worth pursuing include the reversal symmetry in the emitted fields, crucial for the quantum networks. Conclusions and outlook.

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