Resolving Wave Digital Filters with Multiple/Multiport Nonlinearities


Published in:
Proceedings of the 18th International Conference on Digital Audio Effects

Document Version:
Publisher's PDF, also known as Version of record

Queen's University Belfast - Research Portal:
Link to publication record in Queen's University Belfast Research Portal

Publisher rights
Copyright 2015 The Authors
Published in the Proceedings of the 19th International Conference on Digital Audio Effects (DAFx-16)

General rights
Copyright for the publications made accessible via the Queen's University Belfast Research Portal is retained by the author(s) and / or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights.

Take down policy
The Research Portal is Queen's institutional repository that provides access to Queen's research output. Every effort has been made to ensure that content in the Research Portal does not infringe any person's rights, or applicable UK laws. If you discover content in the Research Portal that you believe breaches copyright or violates any law, please contact openaccess@qub.ac.uk.
RESOLVING WAVE DIGITAL FILTERS WITH MULTIPLE/MULTIPORT NONLINEARITIES

Kurt James Werner, Vaibhav Nangia, Julius O. Smith III, Jonathan S. Abel

Center for Computer Research in Music and Acoustics (CCRMA), Stanford University
660 Lomita Drive, Stanford, CA 94305, USA
[kwerner|vnangia|jos|abel]@ccrma.stanford.edu

ABSTRACT

We present a novel framework for developing Wave Digital Filter (WDF) models from reference circuits with multiple/multiport nonlinearities. Collecting all nonlinearities into a vector at the root of a WDF tree bypasses the traditional WDF limitation to a single nonlinearity. The resulting system has a complicated scattering relationship between the nonlinearity ports and the ports of the rest of the (linear) circuit, which can be solved by a Modified-Nodal-Analysis-derived method. For computability reasons, the scattering and vector nonlinearity must be solved jointly; we suggest a derivative of the K-method. This novel framework significantly expands the class of appropriate WDF reference circuits. A case study on a clipping stage from the Big Muff Pi distortion pedal involves both a transistor and a diode pair. Since it is intractable with standard WDF methods, its successful simulation demonstrates the usefulness of the novel framework.

1. INTRODUCTION

The Wave Digital Filter (WDF) concept [1] has been used extensively in physical modeling [2] and virtual analog [3,5]. Researchers in these fields aim to create digital simulations that mimic the physics of reference systems such as guitar amplifiers and effect pedals. Musical circuits of interest may have many nonlinearities (diodes, transistors, triodes, etc.) to which desirable sonic qualities are commonly ascribed [6]. Since WDFs natively support only one nonlinear (NL) circuit element [7], the class of reference circuits which can be modeled by WDFs is very limited.

Though nonlinearity handling in WDFs is an active research area [7,27], creating a WDF from any reference circuit is not a solved problem. Known WDF techniques do not accommodate circuits with multiple/multiport nonlinearities in general.

In this work, we focus on expanding the range of tractable nonlinear reference circuits to include circuits with multiple/multiport nonlinearities. We present a novel framework which is general enough to accommodate circuits with any topology and any number of nonlinearities with any number of ports each. Building on the work of Fränken et al. [43,40], we emphasize topological aspects of the problem and our solution. Rather than treating special cases, this framework treats the common topological issues surrounding multiple and multiport nonlinearities as the norm.

In §2, previous work on WDF nonlinearities is discussed. Our framework is presented in §3, §4, §5, §6, §7, a review of the procedure for deriving system matrices is given in §5. A case study is given in §7. §8 concludes, discusses limitations of our treatment, and speculates about future research directions.

Footnote 1: This perspective is given in abbreviated form in [46].

2. PREVIOUS WORK

The first generation of WDF research [1,50,54] was concerned primarily with creating digital filter structures from analog prototypes, hence leveraging existing analog filter design principles. In 1989, Meerkötter and Scholz noticed that the remaining degree of freedom in a linear WDF could be applied towards including a single algebraic nonlinearity, presenting case studies involving an ideal diode and a piecewise linear resistor [7]. Felderhoff expanded this treatment to reference circuits with a single nonlinear capacitor or inductor by considering WDF counterparts to mutators from classical circuit theory [8,9]. This thread was continued by Sarti and De Poli, who formalized nonlinear reactance handling and studied adaptors “with memory” [10,11]. At the same time, WDFs were being used in physical models of mechanical systems—De Sanctis et al. studied nonlinear frictions and stiffnesses [12] while Pedersini et al. [13,14] and Bilbao et al. [15] studied nonlinear musical applications such as piano hammers and reeds. Under the umbrella of block-based modeling, WDFs found continued application in modeling parts of mechanical systems [16,17,18,24,26].

Until this point, WDFs were limited to a single one-port nonlinearity. When modeling distributed systems in the closely-related digital waveguide context [2], some may consider this to only be a mild restriction, since in that context multiple nonlinearities may sometimes be considered decoupled by propagation delays [30]. However, for lumped systems, the restriction to a single nonlinearity is a significant limitation for WDFs. Attempts to circumvent this limitation have generated a large body of research.

The simplest technique combines multiple nonlinear elements into a single nonlinear one-port. Yeh and Smith derived an implicit diode clipper pair wave domain characteristic via numerical methods [28,29]. Paiva et al. derived a simplified explicit diode pair wave-domain description using the Lambert W function, considering also semi-explicit versions [31,32]. Werner et al. refined aspects of this model and generalized it to diode clippers with an arbitrary number of diodes in each direction, an arrangement common in “modded” and stock guitar distortion pedals [45].

Multiport nonlinear elements can sometimes be simplified to cross-control models [21,22,31,33]. Karjalainen and Pakarinen’s WDF tube amplifier stage assumed zero grid current. This physically reasonable simplification allows the nonlinearity to be modeled as a plate–cathode one-port with the grid voltage as a cross control [21]. They later refined their model by relaxing this assumption [23]. In these circuits and especially those with more than one multiport nonlinear element, the judicious use of ad hoc unit delays aids realizability at the cost of accuracy [29,22,31,33]. This mirrors how unit delays were used to separate stages [30,31].
before the development of reflection-free ports \[54\].

Iterative schemes also hold promise. D’Angelo et al. built on \[21,22,50\] by modeling the triode as a single memoryless nonlinear 3-port WDF element—they designed a secant-method-derived solver, customized to one set of triode equations: the Cardarilli et al. tube model \[37,38\]. Building on their previous work on topology-related delay-free loops \[39\], Schwerdtfeger and Kummet proposed a “multidimensional” approach to global iterative methods which leverages WDF contractivity properties to guarantee convergence in circuits with multiple nonlinearities \[40\].

Nonlinear elements in circuits with only small perturbations around an operating point may be approximated by linearized models. De Sanctis and Sarti suggest a hybrid-$\pi$ bipolar junction transistor (BJT) model for modeling a class-A amplifier \[41\]. A companion paper to the present work discusses how linear controlled sources may be incorporated into WDFs and should be able to accommodate linearization techniques in general \[47\]. Bernardini et al. expanded piecewise linear (PWL) models \[7\] to cases with multiple one-port nonlinearities, yielding realizable structures from algebraic manipulation of a PWL description \[42,43\].

Reference circuits with multiple/multiport nonlinearities tend to have complicated topologies which make them intractable with known WDF techniques. In this research thread, the interaction between multiport nonlinearities and topological issues has been understood, though special cases have been noticed. In 2004, Petrausch and Rabenstein proposed a technique for modeling reference circuits having multiple nonlinearities \[19,24\]; however its application is limited to circuits with a parallel connection between the nonlinearities and a linear subcircuit. Karjalainen and Pakarinen mention computability issues that could arise from splitting WDF structure into pieces that are more approachable, we propose a modification of the SPQR tree of Fränken et al.—this modification involves pulling all of the nonlinear circuit elements out of the root $R$-type adaptor and placing them above the $R$-type adaptor.

Hence the vector nonlinearity is separated from the scattering. With both scattering and the nonlinear elements described in the wave domain, this problem framework is summarized by

$$\begin{align*}
\text{wave nonlinearity} & \quad \{ a_I = f (b_I) \} \quad (1) \\
\text{scattering} & \quad \begin{cases} 
\{ b_I = S_{11} a_I + S_{12} a_E \} & (2) \\
\{ b_E = S_{21} a_I + S_{22} a_E \} & (3)
\end{cases}
\end{align*}$$

with external incident and reflected wave vectors $a_E$ and $b_E$, internal incident and reflected wave vectors $a_I$ and $b_I$, scattering matrix $S = [ S_{11}, S_{12}, S_{21}, S_{22} ]$, and vector nonlinear function $f(\cdot)$.

This is shown as a vector signal flow graph in Fig. 12.

### 4. LOOP RESOLUTION

There are two issues with \[1\]–\[3\] / Fig. 1a. First, they typically form an implicit and coupled set of transcendental equations—something that will not be convenient to solve. Second, there is a noncomputable delay-free loop through $f(\cdot)$ and $S_{11}$. This combination of an $R$-type scattering matrix and a vector of nonlinearities at the root can potentially be handled in various ways. In this paper, we propose a special case of the K-method, a technique for resolving implicit loops in nonlinear state space (NLSS) systems which is well-known in physical modeling \[57,59\].

Our main framework separates \[1\]–\[3\] / Fig. 1a into three phenomena: scattering ($S$ and its partitions) between wave variables at external ($a_E$, $b_E$) and internal ($a_I$, $b_I$) ports, conversion (C and its partitions) between internal wave ($a_C$, $b_C$) and Kirchhoff ($v_C$, $i_C$) variables, and a vector nonlinear Kirchhoff domain relationship $h(\cdot)$ between $v_C$ and $i_C$. This problem framework is

\footnote{A review of their method is given in \[47\].}
summarized by

Kirchhoff nonlinearity
\[
\begin{align*}
    \mathbf{i}_C &= h(\mathbf{v}_C) \quad (4) \\
    \mathbf{v}_C &= \mathbf{C}_{12} \mathbf{i}_C + \mathbf{C}_{11} \mathbf{a}_C \quad (5) \\
    \mathbf{b}_C &= \mathbf{C}_{21} \mathbf{i}_C + \mathbf{C}_{22} \mathbf{a}_C \quad (6) \\
    \mathbf{a}_C &= \mathbf{b}_I \quad (7) \\
    \mathbf{b}_I &= \mathbf{S}_{11} \mathbf{a}_I + \mathbf{S}_{12} \mathbf{a}_E \quad (9) \\
    \mathbf{b}_E &= \mathbf{S}_{21} \mathbf{a}_I + \mathbf{S}_{22} \mathbf{a}_E \quad (10)
\end{align*}
\]

and shown as a vector signal flow graph in Fig. 1b.

In this form, the structure is still noncomputable due to the presence of multiple delay-free loops. To ameliorate this, we consolidate the eight S and C partitions into four matrices E, F, M, and N. First consider (5) and (10) in matrix form,

\[
\begin{bmatrix} \mathbf{v}_C \\ \mathbf{b}_C \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{12} & 0 \\ 0 & \mathbf{S}_{21} \end{bmatrix} \begin{bmatrix} \mathbf{a}_C \\ \mathbf{a}_I \end{bmatrix} + \begin{bmatrix} 0 & \mathbf{C}_{11} \\ \mathbf{S}_{22} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{i}_C \\ \mathbf{a}_E \end{bmatrix},
\]

and (6)–(9) in matrix form (eliminating \( \mathbf{b}_I \) and \( \mathbf{b}_C \)),

\[
\begin{bmatrix} \mathbf{a}_C \\ \mathbf{a}_I \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{11} \mathbf{H}_C & \mathbf{S}_{12} + \mathbf{S}_{11} \mathbf{H}_C \mathbf{S}_{22} \mathbf{S}_{12} \\ \mathbf{H}_C \mathbf{S}_{21} & \mathbf{H}_C \mathbf{S}_{22} \mathbf{S}_{12} \end{bmatrix} \begin{bmatrix} \mathbf{i}_C \\ \mathbf{a}_E \end{bmatrix},
\]

where \( \mathbf{H} = (\mathbf{I} - \mathbf{C}_{22} \mathbf{S}_{11})^{-1} \). Plugging (13) into (11) and recalling (4) yields a consolidated version of (4)–(10),

\[
\begin{align*}
    \mathbf{i}_C &= h(\mathbf{v}_C) \\
    \mathbf{v}_C &= \mathbf{E} \mathbf{a}_E + \mathbf{F} \mathbf{i}_C \\
    \mathbf{b}_I &= \mathbf{M} \mathbf{a}_E + \mathbf{N} \mathbf{i}_C
\end{align*}
\]

with

\[
\begin{align*}
    \mathbf{E} &= \mathbf{C}_{12}(\mathbf{I} + \mathbf{S}_{11} \mathbf{H}_C \mathbf{S}_{22}) \mathbf{S}_{12} \\
    \mathbf{F} &= \mathbf{C}_{12} \mathbf{S}_{11} \mathbf{H}_C \mathbf{S}_{21} + \mathbf{C}_{11} \\
    \mathbf{M} &= \mathbf{S}_{21} \mathbf{H}_C \mathbf{S}_{22} \mathbf{S}_{12} + \mathbf{S}_{22} \\
    \mathbf{N} &= \mathbf{S}_{21} \mathbf{H}_C \mathbf{S}_{21},
\end{align*}
\]

which is shown as a vector signal flow graph in Fig. 1c.

In this form, the structure is still noncomputable due to one remaining delay-free loop. But, since it is now in a standard NLSS form (one without states), the K-method can be used to render it computable. This yields

\[
\begin{align*}
    \mathbf{i}_C &= g(\mathbf{p}) \\
    \mathbf{p} &= \mathbf{E} \mathbf{a}_E + \mathbf{F} \mathbf{i}_C \\
    \mathbf{b}_I &= \mathbf{M} \mathbf{a}_E + \mathbf{N} \mathbf{i}_C
\end{align*}
\]

where \( h(\cdot) : \mathbf{v}_C \to \mathbf{i}_C \) and \( g(\cdot) : \mathbf{p} \to \mathbf{i}_C \) are related by

\[
\begin{bmatrix} \mathbf{p} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & -\mathbf{K} \end{bmatrix} \begin{bmatrix} \mathbf{v}_C \\ \mathbf{i}_C \end{bmatrix}.
\]

In general the loop resolution matrix \( \mathbf{K} \) (the namesake of the K-method) depends on the chosen discretization method [57]. With no dynamics, we simply have \( \mathbf{K} = \mathbf{F} \) from (14). The explicit framework (15) is shown as a vector signal flow graph in Fig. 1d.

Since each entry in \( \mathbf{p} \) is formed by different linear combination of voltages and currents, we can consider \( \mathbf{p} \) to be a pseudo-wave variable, albeit one without an immediate physical interpretation [60].

The transformation (16) allows us to tabulate solutions to \( h(\cdot) \), which is explicit and easy to tabulate, and transform it to \( g(\cdot) \), a
domain where it would be difficult to tabulate directly. The general framework for a reference circuit with multiple/multiport nonlinearities is hence rendered tractable.

5. DERIVING SYSTEM MATRICES

Recalling (15–16) / Fig. 1 requires C and S matrices that characterize the system: we briefly review their derivation.

C follows directly from the standard WDF voltage wave definition. Recalling (15–16) / Fig. 12, C defines the relationship among the voltages $v_C$, currents $i_C$, incident waves $a_C$, and reflected waves $b_C$ at the internal ports. Accordingly, the values of these matrices come from wave variable definitions. Rearranging the standard voltage wave definition [11] yields

$$ a_C = v_C + R_I i_C $$

$$ b_C = v_C - R_I i_C $$

(17)

where $R_I$ is a diagonal matrix of internal port resistances. Hence:

$$ C = \begin{bmatrix}
C_{11} & C_{12} \\
C_{21} & C_{22}
\end{bmatrix}
= \begin{bmatrix}
-R_I & 1 \\
-2R_I & 1
\end{bmatrix}. $$

(18)

S matrices are found using a method which leverages Modified Nodal Analysis on an equivalent circuit.

Recalling (17–18) / Fig. 12, S defines how incident waves $a_E$ and $a_I$ scatter to yield reflected waves $b_E$ and $b_I$; hence S is derived from the topology around the root. A derivation for S was first presented in [27]. We briefly review the derivation here—it is discussed in detail in a companion paper [27].

First, an instantaneous Thévenin port equivalent to the $R$-type topology (which may include multiport linear elements) is formed; each port n is replaced by a resistive voltage source with value $a_n$ and resistance $R_n$ (equal to the port resistance). This circuit is described by Modified Nodal Analysis as

$$ \begin{bmatrix}
Y & A \\
B & D
\end{bmatrix}
\begin{bmatrix}
v_I \\
i_E
\end{bmatrix}
= \begin{bmatrix}
i_L \\
i_E
\end{bmatrix}, $$

(19)

where X partitions Y, A, B, and D are found by well-known element stamp procedures [55–56–58–59]. Confronting (19) with wave variable definitions and compatibility requirements yields

$$ S = I + 2 \begin{bmatrix}
0 & R
\end{bmatrix}
X^{-1}
\begin{bmatrix}
0 & I
\end{bmatrix}^T, $$

(20)

where $R = \text{diag}(R_E, R_E)$ is a diagonal matrix of port resistances. $S_{11}, S_{12}, S_{21},$ and $S_{22}$ are found simply by partitioning S.

6. CASE STUDY

As a case study on the methods presented in this paper, we model the first clipping stage from the Big Muff Pi distortion pedal [62]. This circuit has three nonlinear elements: two 1N914 diodes ($D_3$ and $D_4$) and a 2N5089 BJT. We combine the two diodes into a single one-port nonlinearity as in [25–29–24–35–35]. Hence the WDF we derive has three nonlinear ports: the diode pair, the BJT’s $V_{BE}$ junction, and the BJT’s $V_{CE}$ junction.

The names, values, and graph edges of each circuit element are shown in Table 1; coefficients pertaining to the nonlinear behavior of the diodes and transistor are shown in Table 2.

Table 1: Big Muff Pi Circuit Elements.

<table>
<thead>
<tr>
<th>element</th>
<th>value</th>
<th>description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_{th}$</td>
<td>2.92 nA</td>
<td>diode reverse saturation current</td>
</tr>
<tr>
<td>$V_T$</td>
<td>25.85 mV</td>
<td>thermal voltage</td>
</tr>
<tr>
<td>$I_s$</td>
<td>5.911 fA</td>
<td>BJT reverse saturation current</td>
</tr>
<tr>
<td>$\alpha_f$</td>
<td>0.9993</td>
<td>BJT forward current gain</td>
</tr>
<tr>
<td>$\alpha_b$</td>
<td>0.5579</td>
<td>BJT reverse current gain</td>
</tr>
</tbody>
</table>

The derivation of a WDF from the Big Muff Pi clipping stage is shown in Fig. 2 details follow.

The Big Muff Pi first clipping stage circuit is shown in Fig. 2a. Using the method of Fränken et al. [49], a connected graph structure is formed. In this graph (Fig. 2b), each circuit port corresponds to a graph edge, which is assigned an integer index. As described in §3, we extend the replacement graph approach of Fränken et al. to the nonlinear case—all nodes that are connected to a nonlinear element (nodes $d$, $e$, $f$, and $g$) are bundled together using a replacement graph, shown in gray in Fig. 2b.

A standard search for split components yields the graph shown in Fig. 2b. As expected, the replacement graph has no splits and has ended up embedded in an $R$-type topology which cannot be decomposed further into series and parallel connections.

We form a modified SPQR tree by designating the $R$-type topology as the root of a tree and letting the remaining one-ports and adaptors dangle below it (Fig. 2g). Here we break from standard technique, by extracting the three nonlinear ports (the diode pair $D_{3,4}$, $V_{BC}$, and $V_{BE}$, designated with a dark gray background) from the $R$-type adaptor and placing them above the root element (designated by a light gray background).

A WDF tree (Fig. 2e) follows from the modified SPQR tree of Fig. 2g. A rearranged version of Fig. 2a which highlights the derived adaptor structure is shown in Fig. 2e. Note that $V_{th}$ is given a fictitious 1 $\Omega$ source resistance to avoid having a non-root ideal voltage source, which would cause computability issues.

Recall that a WDF adaptor can have only a single “reflection-free port.” Since the $R$-type adaptor would need three reflection-free ports to guaranteed realizability in the standard WDF framework, we make recourse to the techniques presented in §3 to handle the collection of the $R$-type adaptor and the nonlinearities. The rest of the adaptors and one-ports form WDF subtrees which are already computable with traditional techniques.

The behavior of $R$ and the nonlinearities is described by a system in the form of Fig. 12. Here, $-$K converter matrix C partitions are given by the vertex form of standard voltage wave definitions [18]. Scattering matrix S partitions describe the scattering behavior of the root $R$-type adaptor (Fig. 2f) and can be found using our MNA-derived method [29] on its instantaneous Thévenin port equivalent (Fig. 2h). Our matrix of port resistances

Table 2: Big Muff Pi Circuit Elements.
Figure 2: Deriving a WDF adaptor structure for the Big Muff Pi clipping stage.
is $R = \text{diag}(R_A, R_B, R_C)$, where $R_A = \text{diag}(R_{A1}, R_{A2}, R_{A3})$ and $R_E = \text{diag}(R_{D}, R_{E}, R_{F}, R_{G})$ are matrices of internal and external port resistances. $R_E$ is presribed by the port resistances of the WDF subtrees, and the port resistances in $R_A$ are assigned arbitrarily as $R_A = R_B = R_C = 1 \, \text{k}\Omega$; we cannot make all three internal ports reflection-free, and they will get resolved by the K-method anyways, so we don’t even try to adapt any of them.

According to the Shockley ideal diode law, $i_{\text{diodes}} = I_s/d(e^{v_{\text{diodes}}/V_T} - 1) - I_s(e^{-v_{\text{diodes}}/V_T} - 1)$,\footnote{http://www.mathworks.com/help/matlab/ref/scatteredinterpolant-class.html} the behavior of our vector of nonlinearities is described in the Kirchoff domain by $h(\cdot)$, which maps a vector of voltages $v_C = [v_{\text{diodes}}, v_{BC}, v_{HH}]^T$ to a vector of currents $i_C = [i_{\text{diodes}}, -I_C, i_E]^T$ according to the Shockley ideal diode law,

$$i_{\text{diodes}} = I_s/d(e^{v_{\text{diodes}}/V_T} - 1) - I_s(e^{-v_{\text{diodes}}/V_T} - 1), \quad (21)$$

and the Ebers–Moll BJT model,

$$i_C = I_s(e^{v_{BC}/V_T} - 1) - I_s/e^{v_{HH}/V_T} - 1, \quad (22)$$

$$i_E = I_s/e^{v_{HH}/V_T} - 1 - I_s(e^{v_{HH}/V_T} - 1), \quad (23)$$

$h(\cdot)$ is tabulated with a $201 \times 201 \times 201$ grid across the circuit’s operating range of voltages (determined experimentally with SPICE).

Since this structure contains multiple noncomputable loops, it is collapsed down into a NLSS form (Fig. 12), where the values of $E$, $F$, $M$, and $N$ are given as functions of $C_{11}$, $C_{12}$, $C_{21}$, $C_{22}$, $S_{11}$, $S_{12}$, $S_{21}$, and $S_{22}$ by [12]. This structure still contains a noncomputable loop through $h(\cdot)$ and $F$, but since it is a standard NLSS system (albeit one without states), we can render it computable using the K-method [17,19], yielding a system in the form of Fig. 14. Matrices $E$, $M$, and $N$ are already known, and a scattered tabulation $g(\cdot)$ is found from the gridded tabulation $h(\cdot)$, using $K = F$ according to [15].

To explain how the system is computable, let us walk through one time step of the algorithm. All of the WDF subtrees which dangle from $R$ are handled in the normal way, and the collection of $R$ and the nonlinearities are handled according to [15]. At each time step during runtime, these subtrees deliver a length-5 vector of incident waves $a_R = [a_D, a_E, a_F, a_G, a_H]^T$ to $R$. We find length-3 vector $p$ from $a_R$ by $p = E a_R$. Using scattered interpolant methods [14], we find length-3 vector $i_C$ from $p$ by $i_C = g(p)$. Finally the length-5 output vector $b_R$ of the root $R$–nonlinearity collection is found by combining contributions from $a_E$ and $i_C$ by $b_R = M a_E + N i_C$. $b_R$ is propagated down into the standard WDF subtrees and we can advance to the next time step.

Simulation results are shown in Fig. 16. On top is a 15-ms-long guitar input signal $V_a$. The middle panel compares a “ground-truth” SPICE simulation and the results of our WDF algorithm. They are nearly a match, confirming the validity of our novel WDF formulation. The error signal between the two is shown in the bottom panel; small discrepancies can be ascribed to limitations of the $201 \times 201 \times 201$ lookup table, (expected) aliasing in the WDF, linear resampling of the SPICE output to WDF simulation’s constant time grid for comparison, and general numerical concerns.

7. CONCLUSION AND FUTURE WORK

In this paper, we’ve presented a framework for modeling lumped systems (in particular, electronic circuits) with multiple and multport nonlinear elements and arbitrary topologies as Wave Digital Filters. We expand on the graph-theoretic approach of Fränken et al. [18,19], applying it rather to yield an adaptor structure for circuits with multiple/multiport nonlinearities. Replacing the collection of all nonlinear elements with a single replacement graph during the separation-pair-finding process yields a modified SPQR tree, where the root node tends to be an $R$-type adaptor, with standard WDF subtrees hanging below and a vector of nonlinearities hanging above. This confines issues of computability to a minimal subsystem comprised of the $R$-type adaptor and vector of nonlinearities at the root of a WDF tree.
This combination of this $R$-type adaptor and the vector of nonlinearities is noncomputable. To render it computable, we rearrange it so our system of “consolidated multiplies” $E, F, M$, and $N$ is solvable with the K-method\cite{57,59}. The advantage of solving the root collection with the K-method is that tabulation can be done simply in the Kirchhoff domain, where $i_{C} - v_{C}$ relationships are usually explicit and memoryless.\footnote{Nonlinearities “with memory,” e.g., NL inductors/capacitors, can be expressed as instantaneous aspects combined with WDF “mutators”\cite{11}.} A disadvantage is that when gridded tabulation is transformed to the $p - i_{C}$ domain, it will no longer be gridded, limiting interpolation to scattered methods.

We’ve focused on creating a general problem framework with the potential for solution through diverse methods, not only the K-method-derived approach we’ve proposed. Future work should investigate computational efficiency and other methods for solving the localized minimal root subsystem. The internal structure of $S, C, E, F, M$, and $N$ could potentially be exploited for computational saving in future work.

Using iterative techniques to tabulate the $p - i_{C}$ domain directly, rather than transforming an $i_{C} - v_{C}$ tabulation, it should be possible to create a gridded tabulation, opening up the possibility of using gridded interpolant methods. This would require good initial guesses for the iterative solver or the use of, e.g., Newton Homotopy\cite{58,59}. Potentially, a transformed solution could be used as initial guesses for forming a gridded interpolation\cite{54}.

We could consider keeping our root topology $R$ and vector of nonlinearities entirely in the wave domain, as in Fig.\cite{1a}. In this case, a nonlinear wave-domain function $f(\cdot)$ would need to solve $a_{i} = f(b_{i})$. This system could also be resolved with the K-method. However, it is unlikely that a closed-form solution to $f(\cdot)$ will be available, so an explicit solution can’t be tabulated for $f(\cdot)$ and then transformed. With iterative techniques, tabulation in the transformed $f(\cdot)$ domain should be possible.

As another option, the instantaneous Thévenin port equivalent concept could be applied to all external ports of the nonlinear root node topology and vector nonlinearity, creating a Kirchhoff-domain subsystem grafted onto standard WDF subtrees below. As in the scattering matrix derivation\cite{17}, each external port $n$ in the root scattering matrix would be replaced by its instantaneous Thévenin equivalent—a series combination of a voltage source with a value equal to the incident wave $a_{n}$ and a resistor equal to the port resistance $R_{n}$—and then the system could be solved with the K-method or any other Kirchhoff-domain method.

In our main formulation or the others proposed above, one could develop alternate derivations that exchange the roles of currents and voltages, using e.g., $v_{C} = g_{v}^{-1}(i_{C})$. This may be particularly applicable to using iterative methods to tabulate the nonlinearities; Yeh et al. report that iteration on terminal voltages converged faster than iteration on device currents\cite{53}.

Future work should explore these alternate formulations and practical aspects of $N$-dimensional scattered interpolation. Our exposition uses standard voltage wave variables for simplicity of presentation, but for time-varying structures, power waves provide theoretical energetic advantages\cite{15,16,63}. For alternative wave variable definitions,\cite{17,18} will be modified as appropriate.

Energetic concerns in the root subsystem remain unexplored and potentially problematic, e.g., for highly resonant circuits. Previous work investigates energetic aspects of nonlinear table interpolation techniques in WDFs\cite{71}—although it is limited to the one-port case, it could serve as a prototype for future work.

8. REFERENCES

\begin{enumerate}
  \item V. Valimaki et al., \textit{Virtual Analog Effects}, chapter 12, pp. 473–522, John Wiley & Sons, second edition, 2011. Appears in\cite{64}.
  \item V. Valimaki, “Virtual analog modeling,” in \textit{Int. Conf. Digital Audio Effects (DAFx-13)}, Maynooth, Ireland, Sept. 5 2013, keynote.
\end{enumerate}
M. Karjalainen, “BlockCompiler: Efficient simulation of acoustic
S. Bilbao et al., “The wave digital reed: A passive formulation,” in
M. Karjalainen, “Automatic synthesis strategies for object-based
dynamical physical models in musical acoustics,” in Proc. Int. Conf.
M. Karjalainen, “BlockCompiler: Efficient simulation of acoustic
and audio systems,” in Proc. Audio Eng. Soc. (AES) Conv., Amster-
G. De Sanctis et al., “Automatic synthesis strategies for object-based
dynamical physical models in musical acoustics,” in Proc. Int. Conf.
S. Petrausch and R. Rabenstein, “Wave digital filters with multi-
ple nonlinearities,” in Proc. European Signal Processing Conf. (EU-
M. Karjalainen, “Efficient realization of wave digital components for
physical modeling and sound synthesis,” IEEE Trans. Audio, Speech,
M. Karjalainen and J. Pakarinen, “Wave digital simulation of a
vacuum-tube amplifier,” in Proc. IEEE Int. Conf. Acoust., Speech,
J. Pakarinen et al., “Wave digital modeling of the output chain of a
vacuum-tube amplifier,” in Proc. Int. Conf. Digital Audio Effects
(DAFX-09), Como, Italy, Sept. 1–4 2009, pp. 153–156.
J. Pakarinen and M. Karjalainen, “Enhanced wave digital triode
model for real-time tube amplifier emulation,” IEEE Trans. Audio,
S. Petrausch, Block based physical modeling, Ph.D. diss., Friedrich-
R. Rabenstein et al., “Block-based physical modeling for digital sound
R. Rabenstein and S. Petrausch, “Block-based physical modeling with
D. T.-M. Yeh and J. O. Smith III, “Simulating guitar distortion cir-
cuits using wave digital and nonlinear state-space formulations,” in
Proc. Int. Conf. Digital Audio Effects (DAFx-08), Espoo, Finland,
D. T.-M. Yeh, Digital implementation of musical distortion circuits
A. Sarti and G. De Sanctis, “Systematic methods for the implementa-
G. De Sanctis and A. Sarti, “Virtual analog modeling in the wave-
R. C. D. de Paiva et al., “Real-time audio transformer emulation for
P. Raffensperger, “Toward a wave digital filter model of the Fairchild
670 limiter/Universal Univ.-Erlangen-Nürnberg,” in Proc. ISCAS,
R. C. D. Paiva et al., “Emulation of operational amplifiers and diodes
R. C. D. Paiva, Circuit modeling studies related to guitars and audio
processing, Ph.D. diss., Aalto Univ., Espoo, Finland, 2013.
S. D’Angelo and V. Välimäki, “Wave-digital polarity and current
inverters and their application to virtual analog audio processing,” in
S. D’Angelo et al., “New family of wave-digital triode models,”
IEEE Trans. Audio, Speech, Language Process., vol. 21, no. 2,
S. D’Angelo, Virtual Analog Modeling of Nonlinear Musical Cir-
cuits, Ph.D. diss., Aalto University, Espoo, Finland, 2014.
T. Schwerdtfeger and A. Kummert, “A multidimensional signal pro-
cessing approach to wave digital filters with topology-related delay-