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Published in:
Journal of the Mathematical Society of Japan

Document Version:
Peer reviewed version

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A CLASS OF ALMOST $C_0(K)$-C*-ALGEBRAS

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ABSTRACT. We consider in this paper the family of exponential Lie groups $G_{n,\mu}$, whose Lie algebra is an extension of the Heisenberg Lie algebra by the reals and whose quotient group by the centre of the Heisenberg group is an $ax+b$-like group. The C*-algebras of the groups $G_{n,\mu}$ give new examples of almost $C_0(K)$-C*-algebras.

1. Introduction and notations

Let $\mathcal{A}$ be a C*-algebra and $\hat{\mathcal{A}}$ be its unitary spectrum. The C*-algebra $l^\infty(\hat{\mathcal{A}})$ of all bounded operator fields defined over $\hat{\mathcal{A}}$ is given by

$$l^\infty(\hat{\mathcal{A}}) := \{ A = (A(\pi))_{\pi \in \hat{\mathcal{A}}}; \|A\|_\infty := \sup \|A(\pi)\|_{op} < \infty \},$$

where $\mathcal{H}_\pi$ is the Hilbert space on which $\pi$ acts. Let $\mathcal{F}$ be the Fourier transform of $\mathcal{A}$, i.e.,

$$\mathcal{F}(a) := \hat{a} := (\pi(a))_{\pi \in \hat{\mathcal{A}}},$$

for $a \in \mathcal{A}$.

It is an injective, hence isometric, homomorphism from $\mathcal{A}$ into $l^\infty(\hat{\mathcal{A}})$. Hence one can analyze the C*-algebra $\mathcal{A}$ by recognizing the elements of $\mathcal{F}(\mathcal{A})$ inside the (big) C*-algebra $l^\infty(\hat{\mathcal{A}})$.

We know that the unitary spectrum $\hat{C}^*(G)$ of the C*-algebra $C^*(G)$ of a locally compact group $G$ can be identified with the unitary dual $\hat{G}$ of $G$. If $G$ is an exponential Lie group, i.e., if the exponential mapping $\exp : \mathfrak{g} \to G$ from the Lie algebra $\mathfrak{g}$ to its Lie group $G$ is a diffeomorphism, then the Kirillov-Bernat-Vergne-Pukanszky-Ludwig-Leptin theory shows that there is a canonical homeomorphism $K : \mathfrak{g}^*/G \to \hat{G}$ from the space of coadjoint orbits of $G$ in the linear dual space $\mathfrak{g}^*$ onto the unitary dual space $\hat{G}$ of $G$ (see [Lep-Lud] for details and references). In this case, one can therefore identify the unitary spectrum $\hat{C}^*(G)$ of the C*-algebra of an exponential Lie group with the space $\mathfrak{g}^*/G$ of coadjoint orbits of the group $G$.

The C*-algebra of an $ax+b$-like group was characterised in [Lin-Lud] and the C*-algebras of the Heisenberg group and of the threadlike groups were described in [Lu-Tu] as algebras of operator fields defined on the dual spaces of the groups. The method of describing group C*-algebras as algebras of operator fields defined on the dual spaces was first used in [Fell] and [Lee].

In this paper, we consider the exponential solvable Lie group $G_{n,\mu}$, whose Lie algebra is an extension of the Heisenberg Lie algebra $\mathfrak{h}_n$ by the reals, which means that $\mathbb{R}$ acts on $\mathfrak{h}_n$ by a diagonal matrix with real eigenvalues. The quotient group of $G_{n,\mu}$ by the centre of the Heisenberg group is then an $ax+b$-like group, whose C*-algebra has been determined in [Lin-Lud]. Since the orbit structure of exponential groups is well understood (see for instance [Ar-Lu-Sc]), we can write down the spectrum of the group $G_{n,\mu}$ explicitly and determine its topology.

In [ILL] the example of the group $N_{6,28}$ motivated the introduction of a special class of C*-algebras which we called almost $C_0(K)$-C*-algebra, where $K$ is the algebra of all compact operators on some Hilbert space. In Section 2, we recall the definition and the properties of almost $C_0(K)$-C*-algebras. In Section 3 we introduce the family of the $G_{n,\mu}$ groups and describe the space of coadjoint orbits $\mathfrak{g}_{n,\mu}^*/G_{n,\mu}$. We show that the spectrum $\hat{G}_{n,\mu}$ of $G_{n,\mu}$ is a disjoint union of the sets $\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3$, where $\Gamma_0$ is the set of the characters of $G_{n,\mu}$, $\Gamma_1$ and $\Gamma_2$ are the sets of the representations corresponding to the two-dimensional coadjoint orbits of $G_{n,\mu}$, and $\Gamma_3$ is the...
union of the two generic irreducible representations \( \pi_+, \pi_- \) which correspond to the two open orbits. Note that each of the sets \( \Gamma_i \) needs a special treatment. The sets \( \Gamma_1 \) and \( \Gamma_2 \) have been treated in the paper [Lin-Lud]. In Subsection 4.2, we discover the almost \( C_0(K) \) conditions for \( \Gamma_3 \). This is the most intricate part of the paper and the treatment is inspired by the study of the boundary condition for a class of 4-dimensional orbits in [ILL, Subsection 6.3]. At the end (Subsection 4.4), we describe the actual \( C^* \)-algebra of \( G_{n,\mu} \) as an algebra of operator fields and we see that this \( C^* \)-algebra has the structure of an almost \( C_0(K) \)-\( C^* \)-algebra.

2. Almost \( C_0(K) \)-\( C^* \)-algebras

The following definitions were given in [ILL]; for completeness, we recall them here.

**Definition 2.1.** Let \( A \) be a \( C^* \)-algebra and \( \hat{A} \) be the spectrum of \( A \).

1. Suppose there exists a finite increasing family \( S_0 \subset S_1 \subset \ldots \subset S_d = \hat{A} \) of subsets of \( \hat{A} \) such that for \( i = 1, \ldots, d \), the subsets \( \Gamma_0 = S_0 \) and \( \Gamma_i := S_i \setminus S_{i-1} \) are Hausdorff in their relative topologies. Furthermore we assume that for every \( i \in \{0, \ldots, d\} \) there exists a Hilbert space \( H_i \) and a concrete realization \((\pi_\gamma, H_\gamma)\) of \( \gamma \) on the Hilbert space \( H_i \) for every \( \gamma \in \Gamma_i \). Note that the set \( S_0 \) is the collection \( \mathcal{X} \) of all characters of \( A \).

2. For a subset \( S \subset \hat{A} \), denote by \( CB(S) \) the *-algebra of all uniformly bounded operator fields \((\psi(\gamma) \in B(H_\gamma))_{\gamma \in \mathcal{X} \cap S, i=1,\ldots,d}\), which are operator norm continuous on the subsets \( \Gamma_i \cap S \) for every \( i \in \{1, \ldots, d\} \) for which \( \Gamma_i \cap S \neq \emptyset \). We provide the *-algebra \( CB(S) \) with the infinity-norm:

\[
\|\psi\|_S := \sup_{\gamma \in S} \|\psi(\gamma)\|_{op}.
\]

**Definition 2.2.** Let \( \mathcal{H} \) be a Hilbert space and \( K := K(\mathcal{H}) \) be the algebra of all compact operators defined on \( \mathcal{H} \). A \( C^* \)-algebra \( \hat{A} \) is said to be almost \( C_0(K) \) if for every \( a \in \hat{A} \):

1. The mappings \( \gamma \mapsto \mathcal{F}(a)(\gamma) \) are norm continuous on the different sets \( \Gamma_i \), where \( \mathcal{F} : A \rightarrow L^\infty(\hat{A}) \) is the Fourier transform given by

\[
\mathcal{F}(a)(\gamma) = \hat{a}(\gamma) := \pi_\gamma(a) \quad \text{for} \quad \gamma \in \hat{A} \quad \text{and} \quad a \in A.
\]

2. For each \( i = 1, \ldots, d \), we have a sequence \((\sigma_{i,k} : CB(S_{i-1}) \rightarrow CB(S_i))_k \) of linear mappings which are uniformly bounded in \( k \) (and independent of \( a \)) such that

\[
\lim_{k \to \infty} \text{dis} \left( (\sigma_{i,k}(\mathcal{F}(a)_{|S_{i-1}}) - \mathcal{F}(a)_{|\Gamma_i}, C_0(\Gamma_i, K(\mathcal{H}_i))) \right) = 0,
\]

and

\[
\lim_{k \to \infty} \text{dis} \left( (\sigma_{i,k}(\mathcal{F}(a^*_{|S_{i-1}}) - \mathcal{F}(a^*)_{|\Gamma_i}, C_0(\Gamma_i, K(\mathcal{H}_i))) \right) = 0,
\]

where \( C_0(\Gamma_i, K(\mathcal{H}_i)) \) is the space of all continuous mappings \( \varphi : \Gamma_i \rightarrow K(\mathcal{H}_i) \) vanishing at infinity.

**Definition 2.3.** Let \( D^*(A) \) be the set of all operator fields \( \varphi \) defined over \( \hat{A} \) such that

1. The field \( \varphi \) is uniformly bounded, i.e., we have that \( \|\varphi\| := \sup_{\gamma \in \hat{A}} \|\varphi(\gamma)\|_{op} < \infty \).

2. \( \varphi_{|\Gamma_i} \in CB(\Gamma_i) \) for every \( i = 0, 1, \ldots, d \).

3. For every sequence \((\gamma_k)_{k \in \mathbb{N}} \) going to infinity in \( \hat{A} \), we have that \( \lim_{k \to \infty} \|\varphi(\gamma_k)\|_{op} = 0 \).

4. For each \( i = 1, 2, \ldots, d \),

\[
\lim_{k \to \infty} \text{dis} \left( (\sigma_{i,k}(\varphi_{|S_{i-1}}) - \varphi_{|\Gamma_i}, C_0(\Gamma_i, K(\mathcal{H}_i))) \right) = 0
\]

and

\[
\lim_{k \to \infty} \text{dis} \left( (\sigma_{i,k}(\varphi^*_{|S_{i-1}}) - (\varphi_{|\Gamma_i})^*, C_0(\Gamma_i, K(\mathcal{H}_i))) \right) = 0.
\]
We see immediately that if $A$ is almost $C_0(K)$, then for every $a \in A$, the operator field $F(a)$ is contained in the set $D^*(A)$. In fact it turns out that $D^*(A)$ is a $C^*$-subalgebra of $l^\infty(\hat{A})$ and that $A$ is isomorphic to $D^*(A)$.

**Theorem 2.4.** ([ILL, Theorem 2.6]) Let $A$ be a separable $C^*$-algebra which is almost $C_0(K)$. Then the subset $D^*(A)$ of the $C^*$-algebra $l^\infty(\hat{A})$ is a $C^*$-subalgebra which is isomorphic to $A$ under the Fourier transform.

3. The groups $G_{n,\mu}$

Let $n \in \mathbb{N}^*$, $V_n = \mathbb{R}^{2n}$ and denote by $\omega_n$ the canonical non-degenerate skew-symmetric bilinear form on $V_n$. Let

$$\mathfrak{h}_n := V_n \oplus \mathbb{R}.$$ 

Choose a symplectic basis $\mathcal{B} := \{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$ of $V_n$. Let

$$\mathfrak{g}_{n,\mu} := \mathbb{R} \times \mathfrak{h}_n$$ 

and $A = (1,0_{V_n},0), Z = (0,0_{V_n},1) \in \mathfrak{g}_{n,\mu}$. Then $\{A, X_1 \cdots, X_n, Y_1 \cdots, Y_n, Z\}$ is a basis of $\mathfrak{g}_{n,\mu}$. For

$$\mu := \{\lambda_1, \lambda_1', \ldots, \lambda_n, \lambda_n'\} \subset \mathbb{R}$$

with $\lambda_i + \lambda_j' = 2$ for all $i = 1, \ldots, n$, we define the brackets

$$[A, X_i] = \lambda_i X_i, [A, Y_i] = \lambda_i' Y_i, [A, Z] = 2Z$$

for all $i = 1, \ldots, n$,

and

$$[X_i, Y_j] = \delta_{i,j} Z \quad \text{for } i, j = 1, \ldots, n.$$ 

Eventually by exchanging $X_j$ and $Y_j$ and replacing $X_j$ by $-X_j$ we can assume that $\lambda_j' \geq 0$ for all $j$. We then obtain a structure of an exponential solvable Lie algebra on $\mathfrak{g}_{n,\mu}$, and its subalgebra $\mathfrak{h}_n$ is the Heisenberg Lie algebra.

Define the diagonal operator $l_\mu : V_n \to V_n$ by

$$l_\mu(v) := \sum_{i=1}^n \lambda_i v_i X_i + \lambda_i' v_i' Y_i \quad \text{for } v = \sum_{i=1}^n v_i X_i + \sum_{i=1}^n v_i' Y_i \in V_n.$$ 

For $v = \sum_{i=1}^n v_i X_i + v_i' Y_i \in V_n$ and $a \in \mathbb{R}$, we write

$$a \cdot v := \sum_{i=1}^n e^{a \lambda_i} v_i X_i + e^{a \lambda_i'} v_i' Y_i.$$ 

The corresponding simply connected Lie group $G_{n,\mu}$, which is exponential solvable, can be identified with the space $\mathbb{R} \times V_n \times \mathbb{R}$ equipped with the multiplication

$$(3.0.1) \quad (a, v, c) \cdot (a', v', c') := (a + a', (-a') \cdot v + v', e^{-2a'} c + c' + \frac{1}{2} \omega_n((-a') \cdot v, v')).$$

The inner automorphism $\text{Ad} (a, u)$ on $\mathfrak{h}_n$ is given by

$$\text{Ad} (a, u)(0, v, z) = (a, u)(0, v, z)(-a, -(a \cdot u), 0) = (a, 0, 0)(0, u, 0)(0, v, z)(0, -u, 0)(-a, 0, 0) = (a, 0, 0)(0, v, z + \omega_n(u, v))(-a, 0, 0) = (0, a \cdot v, e^{2a} z + e^{2a} \omega_n(u, v)) \quad \text{for } (v, z) \in \mathfrak{h}_n.$$ 

The centre $Z$ of the normal subgroup $H_n := \{0\} \times V_n \times \mathbb{R}$ of $G_{n,\mu}$ is the subset $Z = \exp(\mathbb{R}Z) = \{0\} \times \{0_{V_n}\} \times \mathbb{R}$. Denote by $G_{V_n}$ the quotient group $G_{n,\mu}/Z$ which can be identified with $\mathbb{R} \times V_n$ equipped with the multiplication

$$(s, v) \cdot (t, w) := (s + t, (-t) \cdot v + w).$$
We write $V_\nu = V_0 \oplus V_+ \oplus V_- = V_0 \oplus V_1$, where
\[
V_+ := \text{span}\{X_j, Y_k; \lambda_j > 0, \lambda_k > 0\},
\]
\[
V_- := \text{span}\{X_j, Y_k; \lambda_j < 0\},
\]
\[
V_0 := \text{span}\{X_j, Y_k; \lambda_j = 0, \lambda_k = 0\},
\]
and $V_1 := V_+ \oplus V_-$. Let
\[
\mu_+ := \mu \cap \mathbb{R}^*_+, \mu_- := \mu \cap \mathbb{R}^*_-, \mu_0 := \mu \cap \{0\},
\]
then we can write
\[
V_+ = \sum_{\lambda \in \mu_+} V_{+\lambda} \quad \text{and} \quad V_- = \sum_{\lambda \in \mu_-} V_{-\lambda},
\]
where $V_{+,\lambda}$ and $V_{-,\lambda}$ are the respective eigenspaces of the operator $l_\mu$.

We can also identify $\mathfrak{g}^*_n$ and $\mathbb{R} A^* \oplus V_0^* \oplus \mathbb{R} Z^* \simeq \mathbb{R} \times V_n \times \mathbb{R}$, and then
\[
\langle \text{Ad}^* (a, u)(a^*, v^*, \lambda^*), (0, v, z) \rangle = \langle (a^*, v^*, \lambda^*), \text{Ad} ((a, u)^{-1})(0, v, z) \rangle
\]
\[
= \langle (a^*, v^*, \lambda^*), (0, (-a) \cdot v, e^{-2\lambda} z + e^{-2\lambda} \omega_n(-a \cdot u), v) \rangle
\]
\[
= \langle 0, v^*, (-a) \cdot v, \lambda^* e^{-2\lambda} z + \lambda^* e^{-2\lambda} \omega_n(-a \cdot u), v) \rangle.
\]
Hence
\[
\text{Ad}^* (a, u)(a^*, v^*, \lambda^*)|_{\mathfrak{g}^*_n} = (a^*, (-a) \cdot v^* - \lambda^* e^{-2\lambda}(a \cdot u) \times \omega_n, \lambda^* e^{-2\lambda}).
\]
Here we denote by $u \times \omega_n$ the linear functional on $V_\nu$ as
\[
u \times \omega_n(v) := \omega_n(u, v) \quad \text{for all} \quad v \in V_\nu.
\]
The coadjoint orbit $\Omega_\ell$ of an element $\ell = (a^*, v^*, \lambda^*) \in \mathfrak{g}^*_n$ is given by
\[
\Omega_\ell = \{ (a^* + v^*[([A]_u)] + 2z \lambda^*, (-a) \cdot v^* - \lambda^* e^{-2\lambda}(a \cdot u) \times \omega_n, \lambda^* e^{-2\lambda}) : a, z \in \mathbb{R}, u \in V_\nu \}.
\]
Hence if $\lambda^* \neq 0$ then the corresponding coadjoint orbit is the subset
\[
\Omega_{\lambda^*} = \mathbb{R} \times V_0^* \times \mathbb{R}^*_+ \lambda^*,
\]
where $V_0^*$ is the linear dual space of $V_\nu$. Therefore we have two open coadjoint orbits
\[
\Omega_\varepsilon := \text{Ad}^* (G_{n,\mu}) \ell_\varepsilon = \mathbb{R} \times V_0^* \times \mathbb{R}^*_+ \lambda^* \quad \text{for} \quad \varepsilon \in \{+, -, 0\},
\]
where $\ell_\varepsilon = \varepsilon Z^*$. The other orbits are contained in $Z^*$ with the form
\[
\Omega_{\lambda^*} = \mathbb{R} A^* + \mathbb{R} \cdot v^* \quad \text{for} \quad v^* \in V_0^* \setminus V_0^*.
\]
We can decompose the linear dual space $V_0^*$ of $V_\nu$ into
\[
V_+^* := \{ f \in V_0^* : f(V_- \cup V_0) = \{0\} \},
\]
\[
V_-^* := \{ f \in V_0^* : f(V_+ \cup V_0) = \{0\} \},
\]
\[
V_0^* := \{ f \in V_0^* : f(V_+ \cup V_-) = \{0\} \}.
\]

The following definition was given in [Lin-Lud2].

**Definition 3.1.** Denote by $\| \cdot \|$ the norm on $V_0^*$ coming from the scalar product defined by the basis $\{X_1, \cdots, X_n, Y_1, \cdots, Y_n\}$. For $f_+ = \sum_{\lambda \in \mu_+} f_\lambda \in V_+^*$ and $f_- = \sum_{\lambda \in \mu_-} f_\lambda \in V_-^*$, let
\[
|f_+|_\mu = |f_+| := \max_{\lambda \in \mu_+} \|f_\lambda\|^{1/\lambda} \quad \text{and} \quad |f_-|_\mu = |f_-| := \max_{\lambda \in \mu_-} \|f_\lambda\|^{-1/\lambda}.
\]
Then for $t \in \mathbb{R}$, we have the relation
\[
|t \cdot f_+| = e^t |f_+| \quad \text{and} \quad |t \cdot f_-| = e^{-t} |f_-| \quad \text{for} \quad f_+ \in V_+^*, f_- \in V_-^*.
\]
On $V_0^*$ we shall use the norm coming from the scalar product. This gives us a global gauge on $V_0^*$:
\[
|\langle f_0, f_+, f_- \rangle| := \max\{|f_0|, |f_+|, |f_-|\}.
\]
We denote by $V^*_{gen}$ the open subset of $V^*_+$ consisting of all the $f = (f_0, f_+, f_-) \in V_0^* \times V_+^* \times V_-^*$ for which $f_+ \neq 0$ and $f_- \neq 0$. The subset $V^*_{sin}$ consists of all the $f = (f_0, f_+, f_-)$ for which either $f_+ \neq 0$, $f_- = 0$ or $f_+ = 0$, $f_- \neq 0$. We see that for every $f = (f_0, f_+, f_-) \in V^*_{gen}$ there exists exactly one element $f' = (f_0', f_+', f_-')$ in its $G_{n, \mu}$-orbit such that $|f_+| = |f_-|$. In the same way, for $f = (f_0, f_+, 0)$ (resp. $f = (f_0, 0, f_-)$) $\in V^*_{sin}$, there exists exactly one element $f' = (f_0, f_+', 0)$ (resp. $f' = (f_0, 0, f_-')$) in its $G_{n, \mu}$-orbit for which $|f_+| = 1$ (resp. $|f_-| = 1$).

For $f_+ \in V^*_+ \setminus \{0\}$, let us denote by $r(f_+)$ the unique real number for which the vector $r(f_+) \cdot f_+$ in $V^*_+$ has gauge 1. This means that

$$r(f_+):= -\ln(|f_+|).$$

Similarly, for $f_- \in V^*_- \setminus \{0\}$ we define the number $q(f_-)$ by

$$q(f_-):= \ln(|f_-|)$$

such that $|q(f_+) \cdot f_-| = 1$. Let

$$D = \{(f_0, f_+, f_-) : |f_+| = |f_-| \neq 0\},$$

$$S_+ = \{(f_0, f_+, 0) : |f_+| = 1\}, S_- = \{(f_0, 0, f_-) : |f_-| = 1\},$$

$$S = S_+ \cup S_-.$$ 

The orbit space $\mathfrak{g}^*_{n, \mu}/G_{n, \mu}$ can then be written as the disjoint union $\Gamma$ of the sets

$$\Gamma_0 = \mathbb{R} \times V^*_+ \cdot \text{S all the unitary characters of } G_{n, \mu},$$

$$\Gamma_1 = S \simeq V^*_{sin}/G_{n, \mu},$$

$$\Gamma_2 = D \simeq V^*_{gen}/G_{n, \mu},$$

$$\Gamma_3 = \{\pm, -\} \simeq \{\Omega, \Omega^-\}/G_{n, \mu},$$

in the case where $V^*_{gen} \neq \emptyset$, i.e., $\mu_+ \neq \emptyset$ and $\mu_- \neq \emptyset$. In case $V^*_{gen} = \emptyset$, we have $\Gamma$ as the union of

$$\Gamma_0 = \mathbb{R} \times V^*_+ \cdot \text{S all the unitary characters of } G_{n, \mu},$$

$$\Gamma_1 = S \simeq V^*_{sin}/G_{n, \mu},$$

$$\Gamma_2 = \{\pm, -\} \simeq \{\Omega, \Omega^-\}/G_{n, \mu}.$$ 

In order to simplify notations, we shall treat only the first case in the following, i.e., we shall assume that $V^*_{gen}$ is nonempty. The other case is similar and easier.

The topology of the orbit space $G^*_{V_n}/G_{V_n}$ of the quotient group $G_{n, \mu}/Z$ has been described in [Lin-Lud]. We recall that a sequence $y = (y_k)_k$ is called properly converging if $y$ has limit points and if every cluster point of the sequence is a limit point, i.e., the set of limit points of any subsequence is always the same, indeed, it equals to the set of all limit points of the sequence $y$.

**Theorem 3.2 (Lin-Lud, Theorem 2.33)**

1. A properly converging sequence $(\Omega f_k)_k$ with $f_k = (f_{0,k}, f_{+,k}, f_{-,k}) \in D$ has either a unique limit point $\Omega f$ for some $f \in D$ and then $f = \lim_k f_k$, or $\lim_k (f_{+,k}, f_{-,k}) = 0$ and then the limit set $L$ of the sequence is given by

$$L = \{\Omega f_{(0,0,f_-)}, \Omega f_{(0,0,f_-)}, \mathbb{R}\},$$

where $f_0 = \lim_k f_{0,k}, f_+ = \lim_k f_{+,k}, f_- = \lim_k f_{-,k}, f_+ \in S_+ \text{ and } f_- = \lim_k f_{+,k} \cdot f_{-,k} \in S_-.$

2. A properly converging sequence $(\Omega f_k)$ with $f_k = (f_{0,k}, f_{+,k}, f_{-,k}) \in S$ has the limit set

$$L = \{\Omega f, \mathbb{R}\},$$

where $f = \lim_k f_k \in S$.

**Corollary 3.3.** The orbit $\Omega f$ for $f \in D$ is closed in $\mathfrak{g}^*_{n, \mu}$. The closure of the orbit $\Omega f$ for $f \in S$ is the set $\{\Omega f, \mathbb{R}\}$.

From the description (3.0.2) of the open orbits $\Omega_{\pm}$, $\varepsilon = \pm$, we have the boundary of $\Omega_{\pm}$ as the following.

**Corollary 3.4.** For $\varepsilon \in \{+,-\}$, the boundary of the open orbit $\Omega_{\pm}$ is the subset $\mathbb{R} \times V^*_+ \times \{0\} = Z^* \simeq \mathfrak{g}^*_{V_n}$.
On the other hand, for every coadjoint orbit we can write down a corresponding irreducible representation as an induced representation by using Kirillov’s orbit theory.

(1) Let $P_n = \exp(\sum_{j=1}^{n} \mathbb{R}Y_j + \mathbb{R}Z)$. This is a closed connected normal abelian subgroup of $G_{n,\mu}$. Let also $F_n := \sum_{j=1}^{n} \mathbb{R}X_j$ and $Y_n := \sum_{j=1}^{n} \mathbb{R}Y_j$ in $V_n$ (an abelian subalgebra of $\mathfrak{g}_{n,\mu}$), then $X_n := \exp(F_n)$ and $Y_n = \exp(\mathfrak{g}_n)$ are closed connected abelian subgroups of $G_{n,\mu}$. We have

$$G_{n,\mu} = \exp(\mathbb{R}A) \cdot X_n \cdot P_n = S_n \cdot P_n,$$

where $S_n := \exp(\mathbb{R}A) \cdot X_n$ is a subgroup of $G_{n,\mu}$. The irreducible representations $\pi_{\varepsilon, \mu}$ corresponding to the orbits $\Omega_{\varepsilon, \mu}$ are of the form

$$\pi_{\varepsilon, \mu} := \text{ind}^{G_{n,\mu}}_{P_n} \varepsilon \lambda (\mathfrak{z}).$$

The Hilbert space of $\pi_{\varepsilon, \mu}$ is the $L^2$-space $L^2(G_{n,\mu}/P_n, \chi_{\varepsilon}) \simeq L^2(S_n)$, where $\chi_{\varepsilon}(y, z) := e^{-2\pi i \varepsilon z}$ for $(y, z) \in P_n$. The elements of this space are the measurable functions $\xi : G_{n,\mu} \rightarrow \mathbb{C}$ satisfying the relations

$$\xi(gp) = \chi_{\varepsilon}(p^{-1})\xi(g) \text{ for } g \in G_{n,\mu}, p \in P_n,$$

and

$$\int_{G_{n,\mu}/P_n} |\xi(g)|^2 dg < \infty,$$

where $dg$ is the left invariant measure on $G_{n,\mu}/P_n$. For $F \in L^1(G_{n,\mu})$ and $\xi \in L^2(G_{n,\mu}/P_n)$, we have

$$\pi_{\varepsilon}(F)\xi(s') = \int_{S_n P_n} F(sp)\xi(p^{-1}s^{-1}s')dsdp$$

$$= \int_{S_n P_n} F(s'p)\xi(p^{-1}s')dsdp$$

$$= \int_{S_n P_n} F(s's^{-1}p)\Delta_{S_n}(s^{-1})\xi(p^{-1}s)dsdp$$

$$= \int_{S_n P_n} F(s's^{-1}p)\Delta_{S_n}(s^{-1})\xi(s^{-1}p^{-1}s)dsdp$$

$$= \int_{S_n P_n} F(s's^{-1}p)\Delta_{S_n}(s^{-1})\chi_{\varepsilon}(s^{-1}ps)\xi(s)dsdp$$

$$= \int_{S_n P_n} F(s's^{-1}p)\Delta_{S_n}(s^{-1})e^{-2\pi \text{Ad}^*n(s)\lambda}s)\xi(s)dsdp$$

$$= \int_{S_n P_n} \tilde{F}^{\pi_n}(s's^{-1}; \text{Ad}^*n(s)\lambda^n)\xi(s)\Delta_{S_n}(s^{-1})ds.$$

Here $\tilde{F}^{\pi_n}$ is the partial Fourier transform of $F$ in the direction $P_n$ given by

$$\tilde{F}^{\pi_n}(s; \ell) := \int_{P_n} F(sp)e^{-2\pi \ell \text{log}(p)} dp \text{ for } s \in S_n, \ell \in \mathfrak{p}_n^*.$$

Hence the operator $\pi_{\varepsilon}(F)$ is given by the kernel function

$$F_{\varepsilon}((a', x'), (a, x)) = \tilde{F}^{\pi_n}(a' - a, a \cdot (x' - x); (-\varepsilon e^{-2a}(a \cdot x) \times \omega_n, \varepsilon e^{-2a})e^{\lambda^n}a),$$

where $|\lambda| := \sum_{j=1}^{n} \lambda_j$. In fact the linear functional $\varepsilon e^{-2a}(a \cdot x) \times \omega_n$ is given by

$$\varepsilon e^{-2a} \varepsilon(a \cdot x) \times \omega_n = \varepsilon \sum_{j=1}^{n} e^{(\lambda_j - 2)a}x_j Y_j^* \text{ for } a \in \mathbb{R}, x \in X_n.$$

Therefore,

$$F_{\varepsilon}((a', x'), (a, x)) = \tilde{F}^{\pi_n}(a' - a, a \cdot (x' - x); (-\varepsilon \sum_{j=1}^{n} e^{(\lambda_j - 2)a}x_j Y_j^*, \varepsilon e^{-2a})e^{\lambda^n}a).$$
Proposition 4.3. For every compact subset \( \Omega \), we have the irreducible representation \( \pi_{\nu^*} \) on \( L^2(\mathbb{R}) \) defined by
\[
\pi_{\nu^*} := \text{ind}^{G_{n,m}}_{H_n} \chi_{\nu^*},
\]
where \( H_n := \exp(h_n) \). The kernel function \( F_{\nu^*} \) of the operator \( \pi_{\nu^*}(F), F \in L^1(G_{n,m}) \), is given by
\[
(3.04) \quad F_{\nu^*}(a,b) = \hat{F}^{\nu^*}(a - b, b \cdot \nu^*, 0) \quad \text{for} \quad a, b \in \mathbb{R}.
\]

Finally, for \( (a^*, \nu_0^*) \in \mathbb{R} \times V_0^* \) we have the unitary characters
\[
\chi_{(a^*, \nu_0^*)}(a, c, c) := e^{-2\pi i (a^* a + \nu_0^*(v_0))} \quad \text{for} \quad a, c \in \mathbb{R}, \ r_0 \in V_0, v \in V_1.
\]

Definition 3.5. We denote by \( L^\infty(\Gamma) \) the C*-algebra
\[
L^\infty(\Gamma) = \{ (\phi, \gamma) \in \mathcal{B}(\mathcal{H}_r)_{\gamma \in \Gamma}; \| \phi \| := \sup_{\gamma \in \Gamma} \| \phi(\gamma) \|_{\text{op}} < \infty \}.
\]

The Fourier transform \( \mathcal{F}_{n,m} : C^*(G_{n,m}) \rightarrow L^\infty(\Gamma) \) for \( C^*(G_{n,m}) \) is given by
\[
\mathcal{F}_{n,m}(a)(\epsilon) = \hat{a}(\epsilon) := \pi_{\epsilon}(a) \quad \text{for} \quad \epsilon \in \{+, -\},
\]
\[
\mathcal{F}_{n,m}(a)(f) = \hat{a}(f) := \pi_f(a) \quad \text{for} \quad f \in \mathcal{D} \cup \mathcal{S},
\]
\[
\mathcal{F}_{n,m}(a^*, \nu_0^*) := \chi_{(a^*, \nu_0^*)}(a) \quad \text{for} \quad (a^*, \nu_0^*) \in \mathbb{R} \times V_0^*,
\]
\[
\langle \int_{\mathbb{R} \times V_0 \times \mathbb{R}} F(s, v_0, v_1, z) e^{-i2\pi a^* s} e^{-i2\pi \nu_0^*(v_0)} ds d\nu_0 d\nu_1 d\epsilon
\]
\[
\quad \text{for} \quad F \in L^1(G_{n,m}).
\]

4. The C*-conditions

4.1. The continuity and infinity conditions.

Theorem 4.1. For every \( a \in C^*(G_{n,m}) \), the mapping
\[
\mathcal{S} \cup \mathcal{D} \rightarrow \mathcal{B}(L^2(\mathbb{R})): f \mapsto \hat{a}(f),
\]
is norm continuous. We also have that
\[
\lim_{|f| \to \infty} \| \pi_{\epsilon}(a) \|_{\text{op}} = 0
\]

Proof. See [Lin-Lud, Proposition 4.2]. \( \square \)

4.2. The condition for the open orbits \( \Omega_\gamma \). To understand the case of open orbits, we have to take into account the boundary points of such an orbit. It is well known that for \( a \in C^*(G) \) the operator \( \pi_{\epsilon}(a) \) is compact if and only if \( \pi(a) = 0 \) for every \( \pi \) in the boundary of the representation \( \pi_{\epsilon} \), i.e., if \( \pi_{\epsilon}(a) = 0 \) for every \( \gamma \in \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \). In this subsection we shall give a description of the algebra of operators \( \pi_{\epsilon}(C^*(G_{n,m})) \).

Definition 4.2. For \( k \in \mathbb{Z} \) and \( r \in \mathbb{R} \), let \( I_{r,k} \) be the half-open interval:
\[
I_{r,k} := \{ kr, kr + r \in \mathbb{R} \}.
\]

1. Let \( S_{\delta,1} := \{ (a, x) \in \mathbb{R} \times \mathcal{X}_n; e^{-a} > \delta^3 \} \).
2. Let \( \delta \mapsto r_\delta \in \mathbb{R}_+ \) be such that \( \lim_{\delta \to 0} r_\delta = +\infty \) and \( \lim_{\delta \to 0} e^{mr_\delta s^{1/2}} = 0 \), where \( m := \max\{2 - \lambda_j\} \).
3. For constants \( D = (D_1, \ldots, D_n) \in (\mathbb{R}^+)^n \) and \( k = (k_0, k_1, \ldots, k_n) \in \mathbb{Z}^{n+1} \), let
\[
S_{\delta, D, k} := \{ (a, x_1, \ldots, x_n) \in \mathbb{R} \times \mathcal{X}_n; e^{-a} \leq \delta^3, a \in I_{r_\delta, k_0}, x_j \in I_{D_j \delta x_j e^{(2 - \lambda_j)s_{k_0, k_j}}, j = 1, \ldots, n} \}.
\]

Proposition 4.3. For every compact subset \( K \subseteq \mathbb{R} \times \mathcal{X}_n \) and \( \delta > 0 \) small enough, we have that
\[
KS_{\delta, D, k} \subset \bigcup_{|\epsilon| \leq \delta} S_{\delta, D_{\epsilon, j_0, k_2}} := R_{\delta, D, k_2},
\]
where \( D_{\delta, j_0} = (D_1 e^{-r_\delta (2 - \lambda_1)(j_0)}, \ldots, D_n e^{-r_\delta (2 - \lambda_n)(j_0)}) \in (\mathbb{R}^+)^n \).
Proof. Indeed, there is an \( M > 0 \) such that \( K \subseteq [-M, M]^{n+1} \subseteq \mathbb{R}^{n+1} \). Let \( r_\delta > M \). For \( (s, u) \in K \) and \( (a, x) \in S_{\delta, D, \kappa} \), it follows that
\[
\zeta := (s, u) \cdot (a, x) = (s + a_s, (-a) \cdot u + x),
\]
and \((k_0 + j_0)r_\delta \leq s + a < (k_0 + j_0 + 1)r_\delta\) for some \( k_0 \in \mathbb{Z} \) and \( j_0 \in \{-1, 0, 1\} \). Furthermore
\[
|e^{i\lambda x}u_j| = |u_j|e^{-\overline{a}\lambda (2\lambda + 1)a} \leq Me^{-2a\delta_\epsilon (2\lambda + 1)k_0(\lambda + 1)} \leq D_\delta \delta^2 e^{\epsilon (2\lambda + 1)(\lambda + 1)k_0(\lambda + 1)},
\]
since \( \delta > 0 \) small enough \( Me^{-2a\delta_\epsilon (2\lambda + 1)k_0(\lambda + 1)} < D_\delta \delta^2 \) for every \( j \). Hence
\[
x_j + e^{-i\lambda x} u_j < (k_j + 1)D_j e^{\epsilon (2\lambda + 1)(\lambda + 1)k_0(\lambda + 1)} + e^{-i\lambda x} u_j < (k_j + 2)D_j e^{\epsilon (2\lambda + 1)(\lambda + 1)k_0(\lambda + 1)},
\]
and also
\[
x_j + e^{-i\lambda x} u_j \geq k_j D_j e^{\epsilon (2\lambda + 1)(\lambda + 1)k_0(\lambda + 1)} - e^{-i\lambda x} u_j \geq (k_j - 1)D_j e^{\epsilon (2\lambda + 1)(\lambda + 1)k_0(\lambda + 1)}.
\]
Therefore \( \zeta \) is contained in the set \( R_{\delta, D, \kappa} \). \( \square \)

Remark 4.4.

1. The family of sets \( \{S_{\delta, 1}, S_{\delta, D, \kappa}; \delta > 0, \kappa \in \mathbb{Z}^{n+1}\} \) forms a partition of \( \mathbb{R}^{n+1} \).
2. Denote by \( M_{\delta, 1} \) the multiplication operator in \( L^2(\mathbb{R}^{n+1}) \simeq L^2(G_{\mu, \nu}/P_n, \chi_{\omega}) \) with the characteristic function of the set \( S_{\delta, 1} \). Similarly let \( M_{\delta, D, \kappa} \) be the multiplication operator on \( L^2(G_{\mu, \nu}/P_n, \chi_{\omega}) \) with the characteristic function of the set \( S_{\delta, D, \kappa} \). These multiplication operators are pairwise disjoint orthogonal projections and the sum of them is the identity operator.

Let \( N_{\delta, D, \kappa} \) be the multiplication operator with the characteristic function of the set \( R_{\delta, D, \kappa} \) for \( \delta > 0 \) and \( \kappa \in \mathbb{Z}^{n+1} \). We have the following property of the operator \( N_{\delta, D, \kappa} \).

Proposition 4.5. There exists a constant \( C > 0 \) such that for any bounded linear operator \( L \) on the Hilbert space \( L^2(G_{\mu, \nu}/P_n, \chi_{\omega}) \), we have that
\[
\| \sum_{\kappa \in \mathbb{Z}^{n+1}} N_{\delta, D, \kappa} \circ L \circ M_{\delta, D, \kappa} \|_{op} \leq C \sup_{\kappa} \| N_{\delta, D, \kappa} \circ L \circ M_{\delta, D, \kappa} \|_{op}.
\]

Proof. See Proposition 6.2 and 6.18 in [ILL]. \( \square \)

Definition 4.6. For \( \kappa \in \mathbb{Z}^{n+1} \) and \( \delta > 0 \), let
\[
\ell_{\kappa, \delta} = -\sum_{j=1}^{n} D_j \delta^2 e^{\epsilon (2\lambda + 1)k_0 \kappa_j Y_j^*} \in b_n^*.
\]
Let \( \sigma_{\kappa, \delta} := \text{ind}_{F_{\mu, \nu}} \chi_{\ell_{\kappa, \delta}} \). The Hilbert space of this representation is the space
\[
\mathcal{H}_{\kappa, \delta} = L^2(G_{\mu, \nu}/P_n, \chi_{\ell_{\kappa, \delta}})
\]
and for \( F \in L^1(G_{\mu, \nu}), \xi \in \mathcal{H}_{\kappa, \delta} \) we have that
\[
\sigma_{\kappa, \delta}(F)\xi(a', x') = \int_{\mathbb{R}^n} \tilde{F}_{\mu, \nu}(s^{-1}; \text{Ad}^*(s)\ell_{\kappa, \delta})\xi(s)\Delta_{\delta}(s^{-1})ds.
\]
Hence this operator has a kernel function given by
\[
F_{\kappa, \delta}(a', a, x) = \tilde{F}_{\mu, \nu}(a' - a, a \cdot (x' - x); ((-a) \cdot \ell_{\kappa, \delta}, 0))e^{\lambda |a|}.
\]
Moreover, the representation \( \sigma_{\kappa, \delta} \) is equivalent to the representation
\[
\overline{\sigma}_{\kappa, \delta} := \int_{\mathbb{R}^n} \pi_{f, \kappa, \delta} df,
\]
where
\[
\pi_{f, \kappa, \delta} := \int_{\mathbb{R}^n} \pi_y \xi d\gamma.
\]
and an equivalence is given by
\[ U_{n,k}(\sigma) : L^2(\mathbb{R} \times \mathcal{X}) \equiv L^2(G_{n,\mu}/P_n, \chi_{f_{k}}) \rightarrow \int_{\mathbb{R}^n} L^2(G_{n,\mu}/H_n, \chi_{f_{k}}) df \]
(4.2.1) \[ U_{n,k}(\xi)(f)(g) := \int_{H_n/P_n} \chi_{f_{k}}(h_n)\xi(g h_n)\hat{h}_n \text{ for } g \in G, f \in p_n^\diamond. \]

Let \( C_{S∪D} \) be the C*-algebra of all uniformly bounded continuous mappings from \( S∪D \) into \( B(L^2(\mathbb{R})) \). It follows from Theorem 4.1 that for every \( a \in C^*(G_{n,\mu}) \) we have that \( \hat{a}_{S∪D} \) is contained in \( C_{S∪D} \).

For each \( f = (f_0, f_+, f_-) \in V_n^* \), we denote by \( f_1 \) the unique element in its coadjoint orbit \( \Omega_f \) contained in \( S∪D \). Let \( U_{n,k,\delta}(f) : L^2(G_{n,\mu}/H_n, \chi_{f_{k,\delta}}) \rightarrow L^2(G_{n,\mu}/H_n, \chi_{f_{k,\delta}}) \) be the canonical intertwining operator of \( \pi_{f_{k,\delta}} \) and \( \pi_{(f_{k,\delta})} \).

Formula (4.2.1) allows us to define a representation of the algebra \( C_{S∪D} \) on the space \( L^2(G_{n,\mu}/P_n) \) by
\[ \tau_{n,k,\delta}(\phi) := U_{n,k,\delta}^{-1} \circ \int_{p_n^\diamond} U_{n,k,\delta}(f)^* \circ \phi((f + \delta_{k,\delta})) \circ U_{n,k,\delta}(f) df \circ U_{n,k,\delta}. \]

We have that
\[ \tau_{n,k,\delta}(a) = \tau_{n,k,\delta}(\hat{a}) \text{ for all } a \in C^*(G_{n,\mu}). \]

**Definition 4.7.** For \( \delta > 0, \, k \in \mathbb{Z}^{n+1} \) and \( a \in C^*(G_{n,\mu}) \), let
\[ \sigma_{n,k,\delta}(a) := \tau_{n,k,\delta}(a) \circ M_{\delta,D,k,2}, \]
\[ \sigma_{n,\delta}(a) := \sum_{k \in \mathbb{Z}^{n+1}} N_{\delta,D,k,2} \circ \sigma_{n,k,\delta}(a). \]

**Proposition 4.8.** Let \( a \in C^*(G_{n,\mu}) \) and \( \varepsilon \in \{+,-\} \). Then
\[ \lim_{\delta \to 0} \text{dis}((\pi_{\varepsilon}(a) - \sigma_{n,\delta}(a)), K(L^2(\mathbb{R} \times \mathcal{X}))) = 0. \]

**Proof.** Let \( L^1 \) be the space of all \( F \in L^1(G_{n,\mu}) \) for which the partial Fourier transform \( \hat{F}^{p_\mu}((a,x),(v^*,s)) \) is a C\(^\infty\)-function with compact support on \( S_n \times p_n^\diamond \). Take \( F \in L^1 \) and choose \( C > 0 \) such that \( \hat{F}^{p_n}((a,x),(v^*,s)) = 0 \), whenever \(|a| + \|x|| > C \) or \(|\|v^*\| + |s| > C \). By Proposition 4.3, for \( \delta > 0 \) small enough, we have that
\[ \pi_{\varepsilon}(a) \circ M_{\delta,D,k,2} = N_{\delta,D,k,2} \circ \pi_{\varepsilon}(F) \circ M_{\delta,D,k,2} \]
for every \( k \) and hence
\[ \pi_{\varepsilon}(F) \circ (I - M_{\delta,1}) - \sigma_{n,\delta}(F) = \pi_{\varepsilon}(F) \circ \left( \sum_{k \in \mathbb{Z}^{n+1}} M_{\delta,D,k,2} \circ \sigma_{n,k,\delta}(F) \right) = \sum_{k \in \mathbb{Z}^{n+1}} N_{\delta,D,k,2} \circ \left( \pi_{\varepsilon}(F) - \sigma_{n,k,\delta}(F) \right) \circ M_{\delta,D,k,2}, \]
and the kernel function \( F_{\delta,k} \) of the operator \( a_{\delta,k} := N_{\delta,D,k,2} \circ \left( \pi_{\varepsilon}(F) - \sigma_{n,k,\delta}(F) \right) \circ M_{\delta,D,k,2} \) is therefore given by
\[ F_{\delta,k}(a',x'), (a,x)) = \left( \hat{F}^{p_\mu}(a' - a,a \cdot (x' - x); (-\varepsilon(\sum_{j=1}^n e^{(\lambda_j - 2)^a x_j Y_j^*}) e^{-2a})) \right) \]
\[ -\hat{F}^{p_\mu}(a' - a,a \cdot (x' - x); (-\varepsilon(a \cdot (k_{\delta,k},0)))) \]
\[ e^{(\lambda_j - 2)^a x_j} \cdot 1_{S_n,D,k,2}(a,x) \cdot 1_{R_{\delta,D,k,2}}(a',x') \text{ for } a,a' \in \mathbb{R}, x,x' \in V_n. \]

We see that
\[ e^{(\lambda_j - 2)^a x_j} - e^{-\lambda_j a} D_\delta^2 e^{(2-\lambda_j)k_0 k_j} = e^{-\lambda_j a} (x_j - D_\delta^2 e^{(2-\lambda_j)k_0 k_j}). \]
Therefore,
\begin{equation}
|e^{(\lambda_j - 2)a} x_j - e^{-\lambda_j a} D_j \delta^2 e^{r_0 (2 - \lambda_j) k_0 k_j}| \\
\leq e^{-\lambda_j a} D_j \delta^2 e^{r_0 (2 - \lambda_j) k_0} \\
= D_j \delta^2 e^{r_0 (2 - \lambda_j) (r_0 k_0 - a)} \\
\leq e^{r_0 m D_j \delta^2} \\
\leq \delta.
\end{equation}

Since \( F \in L^1_c \), there exists a continuous function \( \varphi : S_n \to \mathbb{R}_+ \) with compact support such that
\[ |\hat{F}^n(s; \ell) - \hat{F}^n(s; \ell')| \leq \varphi(s) \| \ell - \ell' \| \quad \text{for } \ell, \ell' \in p^n, s \in S_n. \]

Hence for any \((a, x), (a', x') \in S_n\) and any \( \delta > 0 \) small enough,
\[
|F_{a,k}(a', x'), (a, x))| \\
= |\hat{F}^n(a' - a, a \cdot (x' - x); \left( -\varepsilon \sum_{j=1}^n e^{(\lambda_j - 2)a} x_j Y_j^* \right), \varepsilon e^{-2a}) \rangle| \\
- \hat{F}^n(a' - a, a \cdot (x' - x); \left( -\varepsilon (a) \cdot \ell k, 0 \right)) e^{\left| \lambda_k \right| S_{a, k} (a, x) 1_{R, a, k}^2 (a', x')} \\
\leq \varphi(a' - a, a \cdot (x' - x)) \left| \left( -\varepsilon \sum_{j=1}^n e^{(\lambda_j - 2)a} x_j Y_j^* \right), \varepsilon e^{-2a}) \right|^2 + (\varepsilon - a) \cdot \ell, 0 \right| \\
\leq e^{\left| \lambda_k \right| S_{a, k} (a, x) 1_{R, a, k}^2 (a', x')} \\
\leq C_{a} \varphi(a' - a, a \cdot (x' - x)) e^{\left| \lambda_k \right| a}
\]
for some constant \( C > 0 \) independent of \( \delta \) by (4.2.3). Therefore by Young’s inequality we have that
\[ \| a_{F, \delta} \|_{op} \leq C_{\delta} \quad \text{for } k \in \mathbb{Z}^{n+1}, \]
and finally
\[ \| \pi_\varepsilon(F) \circ (I - M_{\delta, 1}) - \sigma_{a, \delta}(F) \|_{op} \leq C' \delta \]
for a new constant \( C' \), by Proposition 4.5.

On the other hand, the operator \( \pi_\varepsilon(F) \circ M_{\delta, 1} \) is compact since
\[
\| \pi_\varepsilon(F) \circ M_{\delta, 1} \|^2 \leq \infty.
\]

Therefore,
\[
\text{dis}((\pi_\varepsilon(F) - \sigma_{a, \delta}(F)), K(L^2(\mathbb{R} \times \mathcal{X}))) \\
\leq \| \pi_\varepsilon(F) \circ (I - M_{\delta, 1}) - \sigma_{a, \delta}(F) \|_{op} \\
\to 0 \quad \text{as } \delta \to 0.
\]

The Proposition follows, since \( L^1_c \) is dense in \( C^*(G_{n, \mu}) \).
4.3. The two-dimensional orbits $\Omega_{\tau}$ and the characters. The $C^*$-algebras of the groups $G_{V_0} = G_{n,\mu}/\mathbb{Z}$ have been determined as algebras of operator fields in [Lin-Lud]. We adapt this result to our present setting of almost $C_0(K)$-$C^*$-algebras.

**Definition 4.9.** For $a \in C^*(G_{n,\mu})$, let $\Phi(a)$ be the element of $C^*(\mathbb{R} \times V_0)$ defined by $\Phi(a)(\theta) := (\chi_a, a)$ for all $\theta \in \mathbb{R} \times V_0$. The mapping $\Phi : C^*(G_{n,\mu}) \to C^*(\mathbb{R} \times V_0)$ is a surjective homomorphism. Let the kernel of $\Phi$ be denoted by $I_K$, then $C^*(G_{n,\mu})/I_K \simeq C^*(\mathbb{R} \times V_0)$. For $\eta \in C_c(G_{n,\mu})$, the element $\Phi(\eta) \in C^*(\mathbb{R} \times V_0)$ is the continuous function with compact support given by

$$\Phi(\eta)(t, v_0) = \int_{V_0 \times \mathbb{R}} \eta(t, v_0, v, s)dvds \quad \text{for} \quad t \in \mathbb{R}, \ v_0 \in V_0.$$

Choose $\zeta \in C_c(V_1 \times \mathbb{R})$ with $\zeta \geq 0$ and $f_{V_1 \times \mathbb{R}} \zeta(v, s)dvds = 1$, define the mapping $\beta : C_c(\mathbb{R} \times V_0) \to C_c(G_{n,\mu}) \subset C^*(G_{n,\mu})$ by

$$\beta(\varphi)(a, v_0, v, s) = \varphi(a, v_0)\zeta(v, s) \quad \text{for} \quad \varphi \in C_c(\mathbb{R} \times V_0), \ s \in \mathbb{R} \text{ and } v \in V_1.$$

It has been shown in [Lin-Lud] that $\beta$ can be extended to a linear mapping bounded by 1 from $C^*(\mathbb{R} \times V_0)$ into $C^*(G_{n,\mu})$, such that for every $\varphi \in C^*(\mathbb{R} \times V_0)$ we have $\Phi(\beta(\varphi)) = \varphi$.

**Definition 4.10.** Let $(\Omega_{f_k})_k (f_k = (f_{k_1}, f_{k_2}) \in \mathcal{D} \text{ for all } k)$ be a properly converging sequence in $\overline{G_{n,\mu}}$, whose limit set contains the orbits $\Omega_{(f_0,0)}$ and $\Omega_{(0,f_\tau)}$. Let $r_k, q_k \in \mathbb{R}$ be such that $|r_k : f_{k_1}| = 1$ and $|q_k : f_{k_2}| = 1$ for $k \in \mathbb{N}$. Then $\lim_k r_k = -\infty$ and $\lim_k q_k = +\infty$. Choose two positive sequences $(\rho_k), (\kappa_k)$ such that $\kappa_k > \rho_k, -r_k < \rho_k$ for all $k \in \mathbb{N}$, $\lim_{k \to \infty} \kappa_k - q_k = +\infty, \lim_{k \to \infty} \rho_k + r_k = +\infty$ and $\lim_{k \to \infty} \kappa_k - q_k = \lim_{k \to \infty} \rho_k + r_k = 0$. We say that the sequences $(\rho_k, \kappa_k)$ are adapted to the sequence $(f_k)_k$.

For $r \in \mathbb{R}$, let $U(r)$ be the unitary operator on $L^2(\mathbb{R})$ defined by

$$U(r)\xi(s) := \xi(s + r) \quad \text{for all} \xi \in L^2(\mathbb{R}) \text{ and } s \in \mathbb{R}.$$

**Definition 4.11.** Let $A = (A(f) \in \mathcal{B}, f \in \Gamma)$ be a field of bounded operators. We say that $A$ satisfies the *generic condition* if for every properly converging sequence $(\pi_{f_k})_k \subset \mathcal{G}_{n,\mu}$ with $f_k \in \mathcal{D}$ for every $k \in \mathbb{N}$, which admits limit points $\pi_{(f_0,0,f_\tau)}, \pi_{(f_0,f_\tau,0)}$ and for every pair of sequences $\pi_{(\rho_k, \kappa_k)}$ adapted to the sequence $(f_k)_k$ we have that

\begin{align*}
(1) & \lim_{k \to \infty} \|U(r_k) \circ A(f_k) \circ U(-r_k) \circ M(\rho_k, +\infty) - A(f_0, f_+, 0) \circ M(\rho_k, +\infty)\|_\text{op} = 0, \\
(2) & \lim_{k \to \infty} \|U(q_k) \circ A(f_k) \circ U(-q_k) \circ M(-\infty, \kappa_k) - A(f_0, 0, f_-) \circ M(-\infty, \kappa_k)\|_\text{op} = 0.
\end{align*}

The following proposition had been proved in [Lin-Lud, Proposition 5.2].

**Proposition 4.12.** For every $a \in C^*(G_{n,\mu})$, the operator field $F(a)$ satisfies the generic condition.

We must show that on $\mathcal{D}$, our $C^*$-algebra satisfies the almost $C_0(K)$ conditions given in Definition 2.2. For $a \in C^*(G_{n,\mu})$ and $f = (f_0, f_+, f_-) \in V_{gen}^*$, we define the operator

$$\sigma_f(a) := U(-r(f)) \circ \pi_{(f_0,f_+,0)}(a) \circ U(r(f)) \circ M(-\infty, \kappa(f)+r(f))$$

$$+ U(-q(f)) \circ \pi_{(f_0,f_-,0)}(a) \circ U(q(f)) \circ M(q(f)-\rho(f), +\infty),$$

where

$$r(f) = -\ln(|f_+|), \ q(f) = \ln(|f_-|),$$

$$\rho(f) = q(f)^{1/3} - r(f), \ \kappa(f) = q(f) - r(f)^{1/3}.$$

We have the following proposition.

**Proposition 4.13.** For all $f \in \mathcal{D}$, the operator field

$$f \mapsto \sigma_F(f)(a) := \pi_f(a) - \sigma_f(a) \quad (a \in C^*(G_{n,\mu}))$$

is contained in $C_0(\mathcal{D}, \mathcal{K}(L^2(\mathbb{R})))$. 

Theorem 4.16. \( \lim_{t \to \infty} C_t \) is compact. Since the mapping \( f \mapsto \pi_{(F_t)} \) is compact and \( M_{(\rho(f))} \) is compact on \( D \) and for every \( f \in D \), the operator \( \pi_{(F_t)} \otimes M_{(\rho(f))} \) is compact.

The function \( F_{(F_t)} \) is of compact support and \( \rho \) is continuous. Hence the mapping \( f \mapsto \pi_{(F_t,\rho(f))} \) is compact on \( D \) and for every \( f \in D \), the operator \( \pi_{(F_t,\rho(f))} \) is compact on \( D \).

The proposition follows from the density of \( C\Phi \).

Proof. Let \( a \in C^{\ast}(G_{n,\mu}) \). We know that \( \pi_{(a)} \) is a compact operator for any \( f \in V_{\text{gen}}, \) that the mapping \( f \mapsto \pi_{(a)}(f) \) is norm continuous and that \( \lim_{f \to \infty} \pi_{(a)}(f) = 0 \) by Corollary 3.2 and Proposition 4.2 in [Lin-Lud]. If \( f \in L^1_c \), then the kernel function \( F_{f,\rho(f)} \) of the operator \( \pi_{(f_0,f_+)} \otimes M_{(\rho(f))} \) is given by

\[
F_{f_0,f_+}(s,t) = \mathcal{F} \Phi(s-t,t \cdot f_+ \mathcal{P}(\rho(f),\infty)(t)).
\]

The function \( F_{f_0,f_+} \) is of compact support and \( \rho \) is continuous. Hence the mapping \( f \mapsto \pi_{(f_0,f_+)} \otimes M_{(\rho(f))} \) is compact on \( D \) and for every \( f \in D \), the operator \( \pi_{(f_0,f_+)} \otimes M_{(\rho(f))} \) is compact.

Since \( \lim_{f \to \infty} \rho(f) \) goes to infinity as \( ||f|| \) goes to infinity, it follows that \( \pi_{(f_0,f_+)} \otimes M_{(\rho(f))} \) is compact on \( D \). The proposition follows from the density of \( L^1_c \) in \( C^{\ast}(G_{n,\mu}) \).

4.4. The \( C^{\ast}\)-algebras of the groups \( G_{n,\mu} \). Let \( \Gamma_i \subseteq G_{n,\mu}/G_{n,\mu} \) be given as in Section 3.5 and \( \Gamma = \cup \Gamma_i \).

Definition 4.14. (1) For \( f \in D \) and \( \phi \in \ell^\infty(\Gamma) \), let

\[
\sigma_f(\phi) := U(-r(f)) \circ \phi(f_0,f_+,0) \circ U(r(f)) \circ M_{-\infty,\infty(f)+r(f)}
\]

\[
+ U(-q(f)) \circ \phi(f_0,f_-,0) \circ U(q(f)) \circ M_{-\infty,\infty(f)-r(f)+\infty}.
\]

(2) Let \( \varphi = (\varphi(f) \in B(f, \Gamma) \) be a field of bounded operators such that the restriction of the field \( \varphi \) to the set of characters \( \Gamma_0 \) is contained in \( C_0(\Gamma_0) \). We get the element \( \varphi(0) \in C^{\ast}(\mathbb{R} \times V_0) \) determined as in Definition 4.9 by the condition \( \gamma(\varphi(0)) = \varphi(\gamma) \) for \( \gamma \in \Gamma_0 \). We can then define as in Definition 4.9 that

\[
\sigma_f(\varphi) := \beta(\varphi(0)) \in B(\ell^2(\mathbb{R})), \quad f \in S.
\]

Definition 4.15. Let \( D^{\ast}(G_{n,\mu}) \) be the subset of \( l^\infty(G_{n,\mu}) \) defined as a set of all the operator fields \( \phi \) defined over \( \widetilde{G_{n,\mu}} \) such that the mappings \( \gamma \mapsto \phi(\gamma) \) are continuous and vanish at infinity on the sets \( \Gamma_0 \) and \( \Gamma_2 \) and such that \( \phi(f) \in K(L^2(\mathbb{R})) \) for all \( f \in D \). Moreover, each \( \phi \) must fulfill the following conditions:

(1) For \( \varepsilon \in (+,-), \)

\[
\lim_{\varepsilon \to 0} \text{dis}((\phi(\varepsilon) - \sigma_{n,\delta}(\phi)), K(L^2(\mathbb{R} \times \chi))) = 0,
\]

and

\[
\lim_{\varepsilon \to 0} \text{dis}(\phi(\varepsilon) - \sigma_{n,\delta}(\phi^\ast)), K(L^2(\mathbb{R} \times \chi))) = 0.
\]

(2) The mappings

\[
D \ni f \mapsto (\phi(f) - \sigma_f(\phi)) \quad \text{and} \quad D \ni f \mapsto (\phi(f)^\ast - \sigma_f(\phi^\ast))
\]

are contained in \( C_0(D, K(L^2(\mathbb{R}))) \).

(3) The mappings

\[
\mathcal{S} \ni f \mapsto (\phi(f) - \sigma_f(\phi)) \quad \text{and} \quad \mathcal{S} \ni f \mapsto (\phi(f)^\ast - \sigma_f(\phi^\ast))
\]

are contained in \( C_0(\mathcal{S}, K(L^2(\mathbb{R}))) \).

Theorem 4.16. The \( C^{\ast}\)-algebra of \( G_{n,\mu} \) is an almost \( C^{\ast}(\mathbb{K})-C^{\ast}\)-algebra. In particular, the Fourier transform maps \( C^{\ast}(G_{n,\mu}) \) onto the subalgebra \( D^{\ast}(G_{n,\mu}) \) of \( l^\infty(\Gamma) \).

Proof. Propositions 4.8 and 4.13 show that the Fourier transform maps \( C^{\ast}(G_{n,\mu}) \) into \( D^{\ast}(G_{n,\mu}) \). The conditions on \( D^{\ast}(G_{n,\mu}) \) imply that \( D^{\ast}(G_{n,\mu}) \) is a closed involutive subspace of \( l^\infty(\Gamma) \). It follows from [ILL] that \( D^{\ast}(G_{n,\mu}) \) is a \( C^{\ast}\)-subalgebra of \( l^\infty(\Gamma) \) and that \( F_{n,\mu}(C^{\ast}(G_{n,\mu})) = D^{\ast}(G_{n,\mu}) \).
A CLASS OF ALMOST $C_0(K)$-$C^*$-ALGEBRAS

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