A class of almost $C_0(K)$-$C^*$-algebras


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A CLASS OF ALMOST \( C_0(\mathcal{K})\)-C*-ALGEBRAS

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Abstract. We consider in this paper the family of exponential Lie groups \( G_{n,\mu} \), whose Lie algebra is an extension of the Heisenberg Lie algebra by the reals and whose quotient group by the centre of the Heisenberg group is an \( ax+b \)-like group. The C*-algebras of the groups \( G_{n,\mu} \) give new examples of almost \( C_0(\mathcal{K})\)-C*-algebras.

1. Introduction and notations

Let \( \mathcal{A} \) be a C*-algebra and \( \hat{\mathcal{A}} \) be its unitary spectrum. The C*-algebra \( l^\infty(\hat{\mathcal{A}}) \) of all bounded operator fields defined over \( \hat{\mathcal{A}} \) is given by

\[
l^\infty(\hat{\mathcal{A}}) := \{ A = (A(\pi))_{\pi \in \hat{\mathcal{A}}}; \| A \|_\infty := \sup_{\pi} \| A(\pi) \|_{\text{op}} < \infty \},
\]

where \( \mathcal{H}_\pi \) is the Hilbert space on which \( \pi \) acts. Let \( \mathcal{F} \) be the Fourier transform of \( \mathcal{A} \), i.e.,

\[
\mathcal{F}(a) := \hat{a} := (\pi(a))_{\pi \in \hat{\mathcal{A}}} \quad \text{for} \quad a \in \mathcal{A}.
\]

It is an injective, hence isometric, homomorphism from \( \mathcal{A} \) into \( l^\infty(\hat{\mathcal{A}}) \). Hence one can analyze the C*-algebra \( \mathcal{A} \) by recognizing the elements of \( \mathcal{F}(\mathcal{A}) \) inside the (big) C*-algebra \( l^\infty(\hat{\mathcal{A}}) \).

We know that the unitary spectrum \( \hat{C}^*(G) \) of the C*-algebra \( C^*(G) \) of a locally compact group \( G \) can be identified with the unitary dual \( \hat{G} \) of \( G \). If \( G \) is an exponential Lie group, i.e., if the exponential mapping \( \exp : \mathfrak{g} \to G \) from the Lie algebra \( \mathfrak{g} \) to its Lie group \( G \) is a diffeomorphism, then the Kirillov-Bernat-Vergne-Pukanszky-Ludwig-Leptin theory shows that there is a canonical homeomorphism \( K : \mathfrak{g}^*/G \to \hat{G} \) from the space of coadjoint orbits of \( G \) in the linear dual space \( \mathfrak{g}^* \) onto the unitary dual space \( \hat{G} \) of \( G \) (see [Lep-Lud] for details and references). In this case, one can therefore identify the unitary spectrum \( \hat{C}^*(G) \) of the C*-algebra of an exponential Lie group with the space \( \mathfrak{g}^*/G \) of coadjoint orbits of the group \( G \).

The C*-algebra of an \( ax+b \)-like group was characterised in [Lin-Lud] and the C*-algebras of the Heisenberg group and of the threadlike groups were described in [Lu-Tu] as algebras of operator fields defined on the dual spaces of the groups. The method of describing group C*-algebras as algebras of operator fields defined on the dual spaces was first used in [Fell] and [Lee].

In this paper, we consider the exponential solvable Lie group \( G_{n,\mu} \), whose Lie algebra is an extension of the Heisenberg Lie algebra \( \mathfrak{h}_n \) by the reals, which means that \( \mathbb{R} \) acts on \( \mathfrak{h}_n \) by a diagonal matrix with real eigenvalues. The quotient group of \( G_{n,\mu} \) by the centre of the Heisenberg group is then an \( ax+b \)-like group, whose C*-algebra has been determined in [Lin-Lud]. Since the orbit structure of exponential groups is well understood (see for instance [Ar-Lu-Sc]), we can write down the spectrum of the group \( G_{n,\mu} \) explicitly and determine its topology.

In [ILL] the example of the group \( N_{6,28} \) motivated the introduction of a special class of C*-algebras which we called almost \( C_0(\mathcal{K})\)-C*-algebras, where \( \mathcal{K} \) is the algebra of all compact operators on some Hilbert space. In Section 2, we recall the definition and the properties of almost \( C_0(\mathcal{K})\)-C*-algebras. In Section 3 we introduce the family of the \( G_{n,\mu} \) groups and describe the space of coadjoint orbits \( \mathfrak{g}^*_n/G_{n,\mu} \). We show that the spectrum \( G_{n,\mu} \) of \( G_{n,\mu} \) is a disjoint union of the sets \( \Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3 \), where \( \Gamma_0 \) is the set of the characters of \( G_{n,\mu} \), \( \Gamma_1 \) and \( \Gamma_2 \) are the sets of the representations corresponding to the two-dimensional coadjoint orbits of \( G_{n,\mu} \), and \( \Gamma_3 \) is the

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union of the two generic irreducible representations $\pi_+, \pi_-$ which correspond to the two open orbits. Note that each of the sets $\Gamma_i$ needs a special treatment. The sets $\Gamma_1$ and $\Gamma_2$ have been treated in the paper [Lin-Lud]. In Subsection 4.2, we discover the almost $C_0(K)$ conditions for $\Gamma_3$. This is the most intricate part of the paper and the treatment is inspired by the study of the boundary condition for a class of 4-dimensional orbits in [ILL, Subsection 6.3]. At the end (Subsection 4.4), we describe the actual C*-algebra of $G_{n,\mu}$ as an algebra of operator fields and we see that this C*-algebra has the structure of an almost $C_0(K)$-C*-algebra.

2. Almost $C_0(K)$-C*-algebras

The following definitions were given in [ILL]; for completeness, we recall them here.

**Definition 2.1.** Let $A$ be a C*-algebra and $\hat{A}$ be the spectrum of $A$.

1. Suppose there exists a finite increasing family $S_0 \subset S_1 \subset \cdots \subset S_d = \hat{A}$ of subsets of $\hat{A}$ such that for $i = 1, \cdots, d$, the subsets $\Gamma_0 = S_0$ and $\Gamma_i := S_i \setminus S_{i-1}$ are Hausdorff in their relative topologies. Furthermore we assume that for every $i \in \{0, \cdots, d\}$ there exists a Hilbert space $H_i$ and a concrete realization $(\pi_i, H_i)$ of $\gamma$ on the Hilbert space $H_i$ for every $\gamma \in \Gamma_i$. Note that the set $S_0$ is the collection $\mathcal{X}$ of all characters of $A$.

2. For a subset $S \subset \hat{A}$, denote by $CB(S)$ the *-algebra of all uniformly bounded operator fields $(\psi(\gamma) \in B(H_i))_{\gamma \in \mathcal{X} \cap S, i = 1, \cdots, d}$ which are operator norm continuous on the subsets $\Gamma_i \cap S$ for every $i \in \{1, \cdots, d\}$ for which $\Gamma_i \cap S \neq \emptyset$. We provide the *-algebra $CB(S)$ with the infinity-norm:

$$\|\psi\|_S := \sup_{\gamma \in S} \|\psi(\gamma)\|_{\text{op}}.$$  

**Definition 2.2.** Let $\mathcal{H}$ be a Hilbert space and $K := K(\mathcal{H})$ be the algebra of all compact operators defined on $\mathcal{H}$. A C*-algebra $A$ is said to be almost $C_0(K)$ if for every $a \in A$:

1. The mappings $\gamma \mapsto \mathcal{F}(a)(\gamma)$ are norm continuous on the different sets $\Gamma_i$, where $\mathcal{F} : A \to L^\infty(\hat{A})$ is the Fourier transform given by

$$\mathcal{F}(a)(\gamma) = \hat{a}(\gamma) := \pi_{\gamma}(a) \quad \text{for} \quad \gamma \in \hat{A} \text{ and } a \in A.$$  

2. For each $i = 1, \cdots, d$, we have a sequence $(\sigma_{i,k} : CB(S_{i-1}) \to CB(S_i))_{k}$ of linear mappings which are uniformly bounded in $k$ (and independent of $a$) such that

$$\lim_{k \to \infty} \text{dis}\left( (\sigma_{i,k}(\mathcal{F}(a)|_{S_{i-1}}) - \mathcal{F}(a)|_{\Gamma_i}), C_0(\Gamma_i, K(\mathcal{H}_i)) \right) = 0,$$

and

$$\lim_{k \to \infty} \text{dis}\left( (\sigma_{i,k}(\mathcal{F}(a)^*|_{S_{i-1}}) - (\mathcal{F}(a)^*)|_{\Gamma_i}), C_0(\Gamma_i, K(\mathcal{H}_i)) \right) = 0,$$

where $C_0(\Gamma_i, K(\mathcal{H}_i))$ is the space of all continuous mappings $\varphi : \Gamma_i \to K(\mathcal{H}_i)$ vanishing at infinity.

**Definition 2.3.** Let $D^*(A)$ be the set of all operator fields $\varphi$ defined over $\hat{A}$ such that

1. The field $\varphi$ is uniformly bounded, i.e., we have that $\|\varphi\| := \sup_{\gamma \in \hat{A}} \|\varphi(\gamma)\|_{\text{op}} < \infty$.

2. $\varphi|_{\Gamma_i} \in CB(\Gamma_i)$ for every $i = 0, 1, \cdots, d$.

3. For every sequence $(\gamma_k)_{k \in \mathbb{N}}$ going to infinity in $\hat{A}$, we have that $\lim_{k \to \infty} \|\varphi(\gamma_k)\|_{\text{op}} = 0$.

4. For each $i = 1, 2, \ldots, d$,

$$\lim_{k \to \infty} \text{dis}\left( (\sigma_{i,k}(\varphi|_{S_{i-1}}) - \varphi|_{\Gamma_i}), C_0(\Gamma_i, K(\mathcal{H}_i)) \right) = 0$$

and

$$\lim_{k \to \infty} \text{dis}\left( (\sigma_{i,k}(\varphi^*|_{S_{i-1}}) - (\varphi|_{\Gamma_i})^*), C_0(\Gamma_i, K(\mathcal{H}_i)) \right) = 0.$$
We see immediately that if $A$ is almost $C_0(K)$, then for every $a \in A$, the operator field $F(a)$ is contained in the set $D^*(A)$. In fact it turns out that $D^*(A)$ is a $C^*$-subalgebra of $l^\infty(\hat A)$ and that $A$ is isomorphic to $D^*(A)$.

**Theorem 2.4.** ([ILL, Theorem 2.6]) Let $A$ be a separable $C^*$-algebra which is almost $C_0(K)$. Then the subset $D^*(A)$ of the $C^*$-algebra $l^\infty(\hat A)$ is a $C^*$-subalgebra which is isomorphic to $A$ under the Fourier transform.

3. The groups $G_{n,\mu}$

Let $n \in \mathbb{N}^+$, $V_n = \mathbb{R}^{2n}$ and denote by $\omega_n$ the canonical non-degenerate skew-symmetric bilinear form on $V_n$. Let

$$\mathfrak{h}_n := V_n \oplus \mathbb{R}.$$  

Choose a symplectic basis $\mathcal{B} := \{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$ of $V_n$. Let

$$\mathfrak{g}_{n,\mu} := \mathbb{R} \times \mathfrak{h}_n$$

and $A = (1, 0_{V_n}, 0), Z = (0, 0_{V_n}, 1) \in \mathfrak{g}_{n,\mu}$.

Then $\{A, X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z\}$ is a basis of $\mathfrak{g}_{n,\mu}$. For

$$\mu := \{\lambda_1, \lambda_1', \ldots, \lambda_n, \lambda_n'\} \subset \mathbb{R}$$

with $\lambda_i + \lambda_i' = 2$ for all $i = 1, \ldots, n$, we define the brackets

$$[A, X_i] = \lambda_i X_i, \quad [A, Y_i] = \lambda_i' Y_i, \quad [A, Z] = 2Z \quad \text{for all} \quad i = 1, \ldots, n,$

and

$$[X_i, Y_j] = \delta_{ij} Z \quad \text{for} \quad i, j = 1, \ldots, n.$$  

Eventually by exchanging $X_j$ and $Y_j$ and replacing $X_j$ by $-X_j$ we can assume that $\lambda_j' \geq 0$ for all $j$. We then obtain a structure of an exponential solvable Lie algebra on $\mathfrak{g}_{n,\mu}$, and its subalgebra $\mathfrak{h}_n$ is the Heisenberg Lie algebra.

Define the diagonal operator $l_\mu : V_n \rightarrow V_n$

$$l_\mu(v) := \sum_{i=1}^n \lambda_i v_i X_i + \lambda_i' v_i' Y_i$$

for $v = \sum_{i=1}^n v_i X_i + v_i' Y_i \in V_n$ and $a \in \mathbb{R}$, we write

$$a \cdot v := \sum_{i=1}^n e^{a \lambda_i} v_i X_i + e^{a \lambda_i'} v_i' Y_i.$$  

The corresponding simply connected Lie group $G_{n,\mu}$, which is exponential solvable, can be identified with the space $\mathbb{R} \times V_n \times \mathbb{R}$ equipped with the multiplication

$$(3.0.1) \quad (a, v, c) \cdot (a', v', c') := (a + a', -a' \cdot v + v', e^{-2a'} c + c' + \frac{1}{2} \omega_n((-a') \cdot v, v')).$$  

The inner automorphism $Ad(a, u)$ on $\mathfrak{h}_n$ is given by

$$Ad(a, u)(0, v, z) = (a, u, 0)(0, v, z)(-a, -(a \cdot u), 0) = (a, 0, 0)(0, u, 0)(0, v, z)(0, -a, 0)(-a, 0, 0) = (a, 0, 0)(0, v, z + \omega_n(u, v))(-a, 0, 0) = (0, a \cdot v, e^{2a} z + e^{2a} \omega_n(u, v)) \quad \text{for} \quad (v, z) \in \mathfrak{h}_n.$$

The centre $Z$ of the normal subgroup $H_n := \{0\} \times V_n \times \mathbb{R}$ of $G_{n,\mu}$ is the subset $Z = \exp(\mathbb{R}Z) = \{0\} \times \{0_{V_n}\} \times \mathbb{R}$. Denote by $G_{n,\mu} / Z$ the quotient group $G_{n,\mu} / Z$ which can be identified with $\mathbb{R} \times V_n$ equipped with the multiplication

$$(s, v) \cdot (t, w) := (s + t, (t - t) \cdot v + w).$$
We write $V_n = V_0 \oplus V_+ \oplus V_- = V_0 \oplus V_1$, where

\[
V_+ := \text{span}\{X_j, Y_k; \lambda_j > 0, \lambda'_k > 0\},
\]

\[
V_- := \text{span}\{X_j; \lambda_j < 0\},
\]

\[
V_0 := \text{span}\{X_j, Y_k; \lambda_j = 0, \lambda'_k = 0\},
\]

and $V_1 := V_+ \oplus V_-$. Let

\[
\mu_+ := \mu \cap \mathbb{R}^*_+, \quad \mu_- := \mu \cap \mathbb{R}^*_-, \quad \mu_0 := \mu \cap \{0\},
\]

then we can write

\[
V_+ = \sum_{\lambda \in \mu_+} V_+^{\lambda} \quad \text{and} \quad V_- = \sum_{\lambda \in \mu_-} V_-^{\lambda},
\]

where $V_+^{\lambda}$ and $V_-^{\lambda}$ are the respective eigenspaces of the operator $l_\mu$.

We can also identify $g^*_{n,m}$ with $\mathbb{R}A^* \oplus V_n^* \oplus \mathbb{R}Z^* \simeq \mathbb{R} \times V_n \times \mathbb{R}$, and then

\[
(\text{Ad}^*(a,u)(a^*, v^*, \lambda^*), (0, v, z)) = (\langle a^*, v^*, \lambda^* \rangle, (0, v, z))
\]

\[
= ((a^*, v^*, \lambda^*), (0, (−a) \cdot v, e^{-2\alpha}z + e^{-2\alpha}\omega_n(−(a \cdot u), v)))
\]

\[
= (0, v^*, (−a) \cdot v + \lambda^* e^{-2\alpha}z + \lambda^* e^{-2\alpha}\omega_n(−(a \cdot u), v)).
\]

Hence

\[
\text{Ad}^*(a,u)(a^*, v^*, \lambda^*)|_{\mathbb{R}n} = (a^*, (−a) \cdot v^* − \lambda^* e^{-2\alpha}(a \cdot u) \times \omega_n, \lambda^* e^{-2\alpha}).
\]

Here we denote by $u \times \omega_n$ the linear functional on $V_n$ as

\[
u \times \omega_n(v) := \omega_n(u, v) \quad \text{for all } v \in V_n.
\]

The coadjoint orbit $\Omega_\ell$ of an element $\ell = (a^*, v^*, \lambda^*) \in g^*_{n,m}$ is given by

\[
\Omega_\ell = \{ (a^* + v^*([A, u]) + 2z\lambda^*, (−a) \cdot v^* − \lambda^* e^{-2\alpha}(a \cdot u) \times \omega_n, \lambda^* e^{-2\alpha}) : a, z \in \mathbb{R}, u \in V_n \}.
\]

Hence if $\lambda^* \neq 0$ then the corresponding coadjoint orbit is the subset

\[
\Omega_{\lambda^*} := \mathbb{R} \times V_n^* \times \mathbb{R}^*_{\lambda^*},
\]

where $V_n^*$ is the linear dual space of $V_n$. Therefore we have two open coadjoint orbits

\[(3.0.2) \quad \Omega_{\varepsilon} := \text{Ad}^*(G_{n,m})\ell_\varepsilon = \mathbb{R} \times V_n^* \times \mathbb{R}^*_{\varepsilon} \quad \text{for } \varepsilon \in \{+, −\},
\]

where $\ell_\varepsilon = \varepsilon Z^\varepsilon$. The other orbits are contained in $Z^\varepsilon$ with the form

\[
\Omega_{\varepsilon^*} = \mathbb{R}A^* + \mathbb{R} \cdot v^* \quad \text{for } v^* \in V_n^* \setminus V_0^*;
\]

or the one point orbits

\[
\{ a^* A^* + v^* \} \quad \text{for } a^* \in \mathbb{R}, v^* \in V_0^*.
\]

We can decompose the linear dual space $V_n^*$ of $V_n$ into

\[
V_n^* := \{ f \in V_n^* : f(V_- \cup V_0) = \{0\} \},
\]

\[
V_n^+ := \{ f \in V_n^* : f(V_+ \cup V_0) = \{0\} \},
\]

\[
V_n^0 := \{ f \in V_n^* : f(V_+ \cup V_-) = \{0\} \}.
\]

The following definition was given in [Lin-Lud2].

**Definition 3.1.** Denote by $|| \cdot ||$ the norm on $V_n^*$ coming from the scalar product defined by the basis $\{X_1, \cdots, X_n, Y_1, \cdots, Y_n\}$. For $f_+ = \sum_{\lambda \in \mu_+} f_\lambda \in V_n^+$ and $f_- = \sum_{\lambda \in \mu_-} f_\lambda \in V_n^-$, let

\[
|f_+|_\mu = |f_+| := \max_{\lambda \in \mu_+} ||f_\lambda||^{2/\lambda} \quad \text{and} \quad |f_-|_\mu = |f_-| := \max_{\lambda \in \mu_-} ||f_\lambda||^{-2/\lambda}.
\]

Then for $t \in \mathbb{R}$, we have the relation

\[(3.0.3) \quad |t \cdot f_+| = e^t |f_+| \quad \text{and} \quad |t \cdot f_-| = e^{-t} |f_-| \quad \text{for } f_+ \in V_n^+, f_- \in V_n^-.
\]

On $V_n^0$ we shall use the norm coming from the scalar product. This gives us a global gauge on $V_n^*$:

\[
\langle (f_0, f_+, f_-) \rangle := \max\{||f_0||, ||f_+||, ||f_-||\}.
\]
We denote by $V_{gen}^*$ the open subset of $V_n^*$ consisting of all the $f = (f_0, f_+, f_-) \in V^*_0 \times V^*_+ \times V^*_-$ for which $f_+ \neq 0$ and $f_- \neq 0$. The subset $V_{sin}^*$ consists of all the $f = (f_0, f_+, f_-)$ for which either $f_+ \neq 0$, $f_- = 0$ or $f_+ = 0$, $f_- \neq 0$. We see that for every $f = (f_0, f_+, f_-) \in V_{gen}^*$ there exists exactly one element $f' = (f_0, f'_+, f'_-)$ in its $G_{n,\mu}$-orbit such that $|f_+'| = |f_-|$. In the same way, for $f = (f_0, f_+, 0)$ (resp. $f = (f_0, 0, f_-)$) in $V_{sin}^*$, there exists exactly one element $f' = (f_0, f'_+, 0)$ (resp. $f' = (f_0, 0, f'_-)$) in its $G_{n,\mu}$-orbit for which $|f'_+| = 1$ (resp. $|f'_-| = 1$).

For $f_+ \in V^*_+ \setminus \{0\}$, let us denote by $r(f_+)$ the unique real number for which the vector $r(f_+):f_+$ in $V^*_+$ has gauge 1. This means that

$$r(f_+) := -\ln(|f_+|).$$

Similarly, for $f_- \in V^*_- \setminus \{0\}$ we define the number $q(f_-)$ by

$$q(f_-) := \ln(|f_-|)$$

such that $|q(f_+):f_+| = 1$. Let

$$\mathcal{D} = \{(f_0, f_+, f_-) : |f_+| = |f_-| \neq 0\},$$

$$S_+ = \{(f_0, f_+, 0) : |f_+| = 1\}, S_- = \{(f_0, 0, f_-) : |f_-| = 1\}, \quad \text{and} \quad S = S_+ \cup S_-.$$ 

The orbit space $g_{n,\mu}^*/G_{n,\mu}$ can then be written as the disjoint union $\Gamma$ of the sets

$$\Gamma_0 = \mathbb{R} \times V^*_0, \quad \text{corresponding to the unitary characters of } G_{n,\mu},$$

$$\Gamma_1 = S \simeq V_{sin}^*/G_{n,\mu},$$

$$\Gamma_2 = \mathcal{D} \simeq V_{gen}^*/G_{n,\mu},$$

$$\Gamma_3 = \{+,-\} \simeq \{\Omega_+, \Omega_-\}/G_{n,\mu},$$

in the case where $V_{gen}^* \neq \emptyset$, i.e., $\mu_+ \neq \emptyset$ and $\mu_- \neq \emptyset$. In case $V_{gen}^* = \emptyset$, we have $\Gamma$ as the union of

$$\Gamma_0 = \mathbb{R} \times V^*_0, \quad \text{corresponding to the unitary characters of } G_{n,\mu},$$

$$\Gamma_1 = S \simeq V_{sin}^*/G_{n,\mu},$$

$$\Gamma_2 = \{+,-\} \simeq \{\Omega_+, \Omega_-\}/G_{n,\mu}.$$ 

In order to simplify notations, we shall treat only the first case in the following, i.e., we shall assume that $V_{gen}^*$ is nonempty. The other case is similar and easier.

The topology of the orbit space $g_{V_n}^*/G_{V_n}$ of the quotient group $G_{n,\mu}/Z$ has been described in [Lin-Lud]. We recall that a sequence $y = (y_k)_k$ is called properly converging if $y$ has limit points and if every cluster point of the sequence is a limit point, i.e., the set of limit points of any subsequence is always the same, indeed, it equals to the set of all limit points of the sequence $y$.

**Theorem 3.2.** ([Lin-Lud, Theorem 2.3])

1. A properly converging sequence $(\Omega_{f_k})_k$ with $f_k = (f_k, 0, f_+, f_-) \in \mathcal{D}$ has either a unique limit point $\Omega_f$ for some $f \in \mathcal{D}$ and then $f = \lim_{k \to \infty} f_k$, or $\lim_k (f_+, f_-) = 0$ and then the limit set $L$ of the sequence is given by

$$L = \{\Omega_{(f_0, f_+, 0)}, \Omega_{(f_0, 0, f_-)}\},$$

where $f_0 = \lim k f_k, f_+ = \lim_k r(f_k_+) \cdot f_k_+ \in S_+$ and $f_- = \lim_k q(f_k_-) \cdot f_k_- \in S_-$. 

2. A properly converging sequence $(\Omega_{f_k})_k$ with $f_k = (f_k, 0, f_+, f_-) \in \mathcal{S}$ has the limit set

$$L = \{\Omega_f, \mathbb{R}\},$$

where $f = \lim_k f_k \in S$.

**Corollary 3.3.** The orbit $\Omega_f$ for $f \in \mathcal{D}$ is closed in $g_{n,\mu}^*$. The closure of the orbit $\Omega_f$ for $f \in \mathcal{S}$ is the set $\{\Omega_f, \mathbb{R}\}$.

From the description (3.0.2) of the open orbits $\Omega_\varepsilon, \varepsilon = \pm$, we have the boundary of $\Omega_\varepsilon$ as the following.

**Corollary 3.4.** For $\varepsilon \in \{+, -, \}$, the boundary of the open orbit $\Omega_\varepsilon$ is the subset $\mathbb{R} \times V^*_0 \times \{0\} = Z^1 \simeq g_{V_0}^*$. 

On the other hand, for every coadjoint orbit we can write down a corresponding irreducible representation as an induced representation by using Kirillov’s orbit theory.

(1) Let $P_n = \exp(\sum_{j=1}^n 2 \mathbb{R} Y_j + \mathbb{R} Z)$. This is a closed connected normal abelian subgroup of $G_{n,\mu}$. Let also $p_n := \sum_{j=1}^n \mathbb{R} X_j$ and $\eta_n := \sum_{j=1}^n \mathbb{R} Y_j \subset V_n$ (an abelian subalgebra of $\mathfrak{g}_{n,\mu}$), then $X_n := \exp(p_n)$ and $Y_n = \exp(\eta_n)$ are closed connected abelian subgroups of $G_{n,\mu}$.

We have

$$G_{n,\mu} = \exp(\mathbb{R} A) \cdot X_n \cdot P_n = S_n \cdot P_n,$$

where $S_n := \exp(\mathbb{R} A) \cdot X_n$ is a subgroup of $G_{n,\mu}$. The irreducible representations $\pi_{\varepsilon, \varepsilon} = \pm$, corresponding to the orbits $\Omega_{\varepsilon}$, are of the form

$$\pi_{\varepsilon} := \text{ind}_{P_n}^{G_{n,\mu}} \chi_{\varepsilon}.$$

The Hilbert space of $\pi_{\varepsilon}$ is the $L^2$-space $L^2(G_{n,\mu}/P_n, \chi_{\varepsilon}) \simeq L^2(S_n)$, where $\chi_{\varepsilon}(y, z) := e^{-12 \pi \varepsilon \varepsilon}$ for $(y, z) \in P_n$. The elements of this space are the measurable functions $\xi : G_{n,\mu} \to \mathbb{C}$ satisfying the relations

$$\xi(gp) = \chi_{\varepsilon}(p^{-1}) \xi(g) \text{ for } g \in G_{n,\mu}, p \in P_n, \text{ and }$$

$$\int_{G_{n,\mu}/P_n} |\xi(g)|^2 dg < \infty,$$

where $dg$ is the left invariant measure on $G_{n,\mu}/P_n$. For $F \in L^1(G_{n,\mu})$ and $\xi \in L^2(G_{n,\mu}/P_n)$, we have

$$\pi_{\varepsilon}(F)\xi(s') = \int_{S_n P_n} F(sp)\xi(p^{-1}s^{-1}s')dsdp$$

$$= \int_{S_n P_n} F(s'p)\xi(p^{-1}s')dsdp$$

$$= \int_{S_n P_n} F(s'p)\Delta_{S_n}(s^{-1})\xi(p^{-1}s)dsdp$$

$$= \int_{S_n P_n} F(s')\Delta_{S_n}(s^{-1})\xi(s)p^{-1}s)dsdp$$

$$= \int_{S_n P_n} F(s')\Delta_{S_n}(s^{-1})\chi_{\varepsilon}(s^{-1})\xi(s)dsdp$$

$$= \int_{S_n P_n} F(s')\Delta_{S_n}(s^{-1})e^{-12\pi \varepsilon \varepsilon \text{Ad}^*(s)l_{\varepsilon}(\log(p))\xi(s)dsdp$$

$$= \int_{S_n} \hat{F}^{(\varepsilon)}(s^{-1})\text{Ad}^{*}(s)l_{\varepsilon}\xi(s)\Delta_{S_n}(s^{-1})ds.$$

Here $\hat{F}^{(\varepsilon)}$ is the partial Fourier transform of $F$ in the direction $P_n$ given by

$$\hat{F}^{(\varepsilon)}(s; \ell) := \int_{P_n} F(sp)e^{-12\pi \varepsilon \ell(\log(p))} dp \text{ for } s \in S_n, \ell \in \mathfrak{p}_n^*.$$

Hence the operator $\pi_{\varepsilon}(F)$ is given by the kernel function

$$F_{\varepsilon}(a', x'), (a, x)) = \hat{F}^{(\varepsilon)}(a' - a, a \cdot (x' - x); (-\varepsilon e^{-2\alpha} (a \cdot x) \times \omega_n, \varepsilon e^{-2\alpha}))e^{[\lambda|a]}$$

where $|\lambda| := \sum_{j=1}^n \lambda_j$. In fact the linear functional $\varepsilon e^{-2\alpha}(a \cdot x) \times \omega_n$ is given by

$$\varepsilon e^{-2\alpha}(a \cdot x) \times \omega_n = \varepsilon \sum_{j=1}^n e^{(\lambda_j - 2)a_j x_j Y_j^*} \text{ for } a \in \mathbb{R}, x \in X_n.$$

Therefore,

$$F_{\varepsilon}(a', x'), (a, x)) = \hat{F}^{(\varepsilon)}(a' - a, a \cdot (x' - x); (-\varepsilon \sum_{j=1}^n e^{(\lambda_j - 2)a_j x_j Y_j^*}, \varepsilon e^{-2\alpha}))e^{[\lambda|a]}.$$
Proposition 4.3. For every compact subset $D$ of $\mathbb{R}$ defined by $D := \text{ind}_{H_n}^* \chi_{v^*}$, we denote by $\pi_{v^*} := \text{ind}_{H_n}^* \chi_{v^*}$, where $H_n := \exp(b_n)$. The kernel function $F_{v^*}$ of the operator $\pi_{v^*}(F), F \in L^1(G_n, \mu)$, is given by

$$F_{v^*}(a, b) = \tilde{F}_{h_n}(a - b, b \cdot v^*, 0) \quad \text{for} \quad a, b \in \mathbb{R}.$$ 

Definition 3.5. The Fourier transform $F$ is norm continuous. We also have that

$$\pi_{v^*} = \pi_{v^*}(a) \quad \text{for} \quad a \in \mathbb{R}.$$ 

Proof. Definition 4.2. the algebra of operators $\pi$, i.e., if $\pi_{v^*}$, $\pi_{v^*} = \pi_{v^*}(a)$, then $\pi_{v^*}(a) = a \pi_{v^*}(a)$ for $a \in \mathbb{R}, v_0 \in V_0, v \in V_1$.

4. The $C^*$-conditions

4.1. The continuity and infinity conditions.

Theorem 4.1. For every $a \in C^*(G_n, \mu)$, the mapping

$$\mathcal{S} \cup \mathcal{D} \rightarrow \mathcal{B}(L^2(\mathbb{R})): f \mapsto \hat{a}(f),$$

is norm continuous. We also have that

$$\lim_{|f| \to \infty} \|\pi_f(a)\|_{op} = 0.$$

Proof. See [Lin-Lud, Proposition 4.2].

4.2. The condition for the open orbits $\Omega_\gamma$. To understand the case of open orbits, we have to take into account the boundary points of such an orbit. It is well known that for $a \in C^*(G)$ the operator $\pi_{x}(a)$ is compact if and only if $\pi(a) = 0$ for every $\pi$ in the boundary of the representation $\pi_x$, i.e., if $\pi_x(a) = 0$ for every $\gamma \in \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$. In this subsection we shall give a description of the algebra of operators $\pi_x(C^*(G_n, \mu))$.

Definition 4.2. For $k \in \mathbb{Z}$ and $r \in \mathbb{R}$, let $I_{r,k}$ be the half-open interval:

$$I_{r,k} := [kr, kr + r] \subset \mathbb{R}.$$ 

1. Let $S_{\delta,1} := \{(a, x) \in \mathcal{R} \times \mathcal{X}_\epsilon; e^{-a} > \delta^3\}$.
2. Let $\delta \rightarrow r_5 \in \mathcal{R}_e$ be such that $\lim_{\delta \to 0} r_5 = +\infty$ and $\lim_{r \to 0} e^{-mr} \delta^{1/2} = 0$, where $1 \leq m := \max_{x_j} (2 - \lambda_j)$.
3. For constants $D = (D_1, \cdots, D_n) \in (\mathbb{R}_e^+)^n$ and $k = (k_0, k_1, \cdots, k_n) \in \mathbb{Z}_e^{n+1}$, let $S_{\delta, D, k, 2} := \{(a, x_1, \cdots, x_n) \in \mathcal{R} \times \mathcal{X}_\epsilon; e^{-a} \leq \delta^3, a \in I_{r_5, k_0}, x_j \in I_{D_j e^{-r_5(2 - \lambda_j)} k_0, k_j}, j = 1, \cdots, n\}$.

Proposition 4.3. For every compact subset $K \subseteq \mathcal{R} \times \mathcal{X}_\epsilon$ and $\delta > 0$ small enough, we have that

$$K \subseteq \bigcup_{J_0 \in \mathbb{Z}} S_{\delta, J_0, 2} := R_{\delta, J_0, 2},$$

where $D_{\delta, J_0} = (D_1 e^{-r_5(2 - \lambda_1)}(J_0), \cdots, D_ne^{-r_5(2 - \lambda_n)}(J_0)) \in (\mathbb{R}_e^+)^n$. 

□
\[ \text{Proof.} \] Indeed, there is an \( M > 0 \) such that \( K \subset [-M, M]^{n+1} \subset \mathbb{R}^{n+1} \). Let \( r_\delta > M \). For \( (s, u) \in K \) and \( (a, x) \in S_{\delta, D, \delta, 2} \), it follows that
\[ \zeta := (s, u) \cdot (a, x) = (s + a, (-a) \cdot u + x), \]
and \( (k_0 + j_0)r_\delta \leq s + a < (k_0 + j_0 + 1)r_\delta \) for some \( k_0 \in \mathbb{Z} \) and \( j_0 \in \{-1, 0, 1\} \). Furthermore
\[ |e^{-a\lambda_j}u_j| = |u_j|e^{-2a\lambda_j} < M e^{-2a\lambda_j}(a \cdot u + x), \]
and also
\[ \langle x \rangle = \langle x \rangle \leq D_j e^{e^{-x_2(2-\lambda_j)}(a \cdot u + x)} + e^{-a\lambda_j}u_j \]
Furthermore, the representation \( \delta \geq \delta \) for \( \zeta \in \mathbb{R}^{n+1} \).

Remark 4.4.

(1) The family of sets \( \{S_{\delta,1}, S_{\delta, D, \delta, 2}; \delta > 0, \delta \in \mathbb{Z}^{n+1}\} \) forms a partition of \( \mathbb{R}^{n+1} \).

(2) Denote by \( M_{\delta,1} \) the multiplication operator in \( L^2(\mathbb{R}^{n+1}) \cong L^2(G_{n, \mu}/P_n, \chi_\delta) \) with the characteristic function of the set \( S_{\delta,1} \). Similarly let \( M_{\delta, D, \delta, 2} \) be the multiplication operator on \( L^2(G_{n, \mu}/P_n, \chi_\delta) \) with the characteristic function of the set \( S_{\delta, D, \delta, 2} \). These multiplication operators are pairwise disjoint orthogonal projections and the sum of them is the identity operator.

Let \( \mathbb{N}_{\delta, D, \delta, 2} \) be the multiplication operator with the characteristic function of the set \( R_{\delta, D, \delta, 2} \) for \( \delta > 0 \) and \( \delta \in \mathbb{Z}^{n+1} \). We have the following property of the operator \( \mathbb{N}_{\delta, D, \delta, 2} \).

Proposition 4.5. There exists a constant \( C > 0 \) such that for any bounded linear operator \( L \) on the Hilbert space \( L^2(G_{n, \mu}/P_n, \chi_\delta) \), we have that
\[ \| \sum_{\delta \in \mathbb{Z}^{n+1}} \mathbb{N}_{\delta, D, \delta, 2} \circ L \circ \mathbb{M}_{\delta, D, \delta, 2} \|_{\text{op}} \leq C \sup_{\delta} \| \mathbb{N}_{\delta, D, \delta, 2} \circ L \circ \mathbb{M}_{\delta, D, \delta, 2} \|_{\text{op}}. \]

Proof. See Proposition 6.2 and 6.18 in [ILL].

Definition 4.6. For \( \delta \in \mathbb{Z}^{n+1} \) and \( \delta > 0 \), let
\[ \ell_{\delta, \delta} = -e \sum_{j=1}^n D_j e^{e^{-x_2(2-\lambda_j)}(a \cdot y_j + x)} \in \mathfrak{h}_n^*. \]

Let \( \sigma_{\delta, \delta} := \text{int}_{\mathfrak{g}_{n, \mu}} \chi_{\delta, \delta} \). The Hilbert space of this representation is the space
\[ \mathcal{H}_{\delta, \delta} = L^2(G_{n, \mu}/P_n, \chi_{\delta, \delta}) \]
and for \( F \in L^1(G_{n, \mu}), \xi \in \mathcal{H}_{\delta, \delta} \) we have that
\[ \sigma_{\delta, \delta}(F)(x, a, x') = \int \mathcal{F}_{\delta, \delta}(a, x') \Delta_{\delta, \delta}(s^{-1}) ds. \]

Hence this operator has a kernel function given by
\[ F_{\delta, \delta}(a, x') = \mathcal{F}_{\delta, \delta}(a - a \cdot (x' - x); ((-a) \cdot \xi, 0)) \xi(s)^{-1}. \]

Moreover, the representation \( \sigma_{\delta, \delta} \) is equivalent to the representation
\[ \sigma_{\delta, \delta} = \int_{\mathbb{R}^n} \pi_f \ell_{\delta, \delta} df, \]
where
and an equivalence is given by

\[ U_{n,k} : L^2(\mathbb{R} \times \mathcal{A}) \cong L^2(G_{n,\mu}/P_n, \chi_{s,k}) \rightarrow \int_{p_n^+} \chi_{f+s,k}(h_n)dh_n \]

(4.2.1)

\[ U_{n,k}(f)(g) := \int_{p_n^+} \chi_{f+s,k}(h_n)\xi(gh_n)dh_n \text{ for } g \in G, f \in p_n^+. \]

Let \( C_{S\cup D} \) be the C*-algebra of all uniformly bounded continuous mappings from \( S \cup D \) into \( \mathcal{B}(L^2(\mathbb{R})) \). It follows from Theorem 4.1 that for every \( a \in C^*(G_{n,\mu}) \) we have that \( \hat{a}_{S\cup D} \) is contained in \( C_{S\cup D} \).

For each \( f = (f_0, f_+, f_-) \in V_n^* \), we denote by \( f_1 \) the unique element in its coadjoint orbit \( \Omega_f \) contained in \( S \cup D \). Let \( U_{n,k}^\delta(f) : L^2(G_{n,\mu}/H_n, \chi_{f+s,k}) \rightarrow L^2(G_{n,\mu}/H_n, \chi_{f+s,k}) \) be the canonical intertwining operator of \( \pi_{f+s,k} \) and \( \pi_{(f+s,k)_1} \). Formula (4.2.1) allows us to define a representation of the algebra \( C_{S\cup D} \) on the space \( L^2(G_{n,\mu}/P_n) \) by

\[ \tau_{n,k}(\phi) := U_{n,k}^{-1} \circ \int_{p_n^+} U_{n,k}^\delta(f)^* \circ \phi((f + \xi_{s,k})) \circ U_{n,k}^\delta(f)df \circ U_{n,k} \cdot \]

We have that

(4.2.2)

\[ \tau_{n,k}(a) = \tau_{n,k}(\hat{a}) \text{ for all } a \in C^*(G_{n,\mu}). \]

**Definition 4.7.** For \( \delta > 0, k \in \mathbb{Z}^{n+1} \) and \( a \in C^*(G_{n,\mu}) \), let

\[ \sigma_{n,k}^\delta(a) := \tau_{n,k}(a) \circ M_{\delta,D,k} \]

\[ \sigma_{n,k}(a) := \sum_{k \in \mathbb{Z}^{n+1}} N_{\delta,D,k} \circ \sigma_{n,k}^\delta(a). \]

**Proposition 4.8.** Let \( a \in C^*(G_{n,\mu}) \) and \( \varepsilon \in \{+, -\} \). Then

\[ \lim_{\delta \rightarrow 0} \text{dis}(\pi_\varepsilon(a) - \sigma_{n,\delta}(a)), K(L^2(\mathbb{R} \times \mathcal{A}))) = 0. \]

**Proof.** Let \( L^1_k \) be the space of all \( F \in L^1(G_{n,\mu}) \) for which the partial Fourier transform \( \hat{F}^{p_n^+}((a, x), (v^*, s)) \) is a \( C^\infty \)-function with compact support on \( S_n \times p_n^+ \). Take \( F \in L^1_k \) and choose \( C > 0 \) such that \( \hat{F}^{p_n^+}((a, x), (v^*, s)) = 0 \), whenever \( |a| + ||x|| > C \) or \( ||v^*|| + |s| > C. \) By Proposition 4.3, for \( \delta > 0 \) small enough, we have that

\[ \pi_\varepsilon(F) \circ M_{\delta,D,k} = N_{\delta,D,k} \circ \pi_\varepsilon(F) \circ M_{\delta,D,k} \]

for every \( k \) and hence

\[ \pi_\varepsilon(F) \circ (1 - M_{\delta,1}) - \sigma_{n,\delta}(F) = \pi_\varepsilon(F) \circ \left( \sum_k M_{\delta,k} \right) - \sigma_{n,\delta}(F) \]

\[ = \sum_{k \in \mathbb{Z}^{n+1}} N_{\delta,D,k} \circ \left( \pi_\varepsilon(F) - \tau_{n,k}(a) \circ M_{\delta,D,k} \right) \]

and the kernel function \( F_{\delta,k} \) of the operator \( a_{\varepsilon,k} := N_{\delta,D,k} \circ \left( \pi_\varepsilon(F) - \tau_{n,k}(a) \circ M_{\delta,D,k} \right) \) is therefore given by

\[ F_{\delta,k}((a', x'), (a, x)) = \left( \hat{F}^{p_n^+}(a' - a, a \cdot (x' - x); (-\varepsilon(n \sum_{j=1}^n e^{(\lambda_j - 2\alpha)Y_j}; \varepsilon e^{-2\alpha})) \right. \]

\[ - \hat{F}^{p_n^+}(a' - a, a \cdot (x' - x); (-\varepsilon(1 + \xi_{s,k}; 0))) \]

\[ e^{(\lambda_j - 2\alpha)1_{S_o,D,k}^0(a, x)1_{R_o,D,k}^0(a', x')} \text{ for } a, a' \in \mathbb{R}, x, x' \in V_n. \]

We see that

\[ e^{(\lambda_j - 2\alpha)Y_j} - e^{-\lambda_j\alpha}D_j\delta^2e^{(2 - \lambda_j)k_0k_j} = e^{-\lambda_j\alpha}(x_j - D_j\delta^2e^{(2 - \lambda_j)k_0k_j}). \]
Hence,
\begin{equation}
(4.2.3)
\begin{aligned}
|e^{(\lambda_j-2)a}x_j - e^{-\lambda_j^a}D_j\delta^2 e^{r_1(2-\lambda_j)k}k_j| \\
&\leq e^{-\lambda_j^a}D_j\delta^2 e^{r_1(2-\lambda_j)k}k_j \\
&= D_j\delta^2 e^{r_1(2-\lambda_j)k}k_j \\
&\leq e^{r_1m}D_j\delta^2 \\
&\leq \delta.
\end{aligned}
\end{equation}

Since $F \in L^1_c$, there exists a continuous function $\varphi : S_n \to \mathbb{R}$ with compact support such that
\[ \left| \hat{F}^{p_n}(s; t) - \hat{F}^{p_n}(s; t') \right| \leq \varphi(s) \| t - t' \| \quad \text{for } t, t' \in p_n^*, s \in S_n. \]

Whence for any $(a, x), (a', x') \in S_n$ and any $\delta > 0$ small enough,
\[ |F_{a, \delta, k}((a', x'), (a, x))| = |\hat{F}^{p_n}(a' - a, a \cdot (x' - x); (-\varepsilon \sum_{j=1}^n e^{(\lambda_j-2)a}x_j Y_j^*), e^{\varepsilon-2a})| - |\hat{F}^{p_n}(a' - a, a \cdot (x' - x); (-\varepsilon (a \cdot \ell_{\delta, \delta}, 0))e^{(\lambda_j-2)a}x_j Y_j^*, e^{\varepsilon-2a})| \leq \varphi(a' - a, a \cdot (x' - x))\|(-\varepsilon \sum_{j=1}^n e^{(\lambda_j-2)a}x_j Y_j^*), e^{\varepsilon-2a}) + (\varepsilon (a \cdot \ell_{\delta, \delta}, 0))\| \]
\[ \leq C\delta \varphi(a' - a, a \cdot (x' - x))e^{\varepsilon}\|a, \delta, k\| \]
for some constant $C > 0$ independent of $\delta$ by (4.2.3). Therefore by Young’s inequality we have that
\[ \|a, \delta, k\| \leq C\delta \quad \text{for } k \in \mathbb{Z}^{n+1}, \]
and finally
\[ \| \pi_{\varepsilon}(F) \circ (I - M_{\delta, 1}) - \sigma_{n, \delta}(F) \| \leq C'\delta \]
for a new constant $C'$, by Proposition 4.5.

On the other hand, the operator $\pi_{\varepsilon}(F) \circ M_{\delta, 1}$ is compact since
\[ \| \pi_{\varepsilon}(F) \circ M_{\delta, 1} \|_{H^{-S}}^2 = \int_{\mathbb{R}} \int_{(-\varepsilon > \delta^3)} \int_{(x \times X)} \left| \hat{F}^{p_n}(a' - a, a \cdot (x' - x); (-\varepsilon \sum_{j=1}^n e^{(\lambda_j-2)a}x_j Y_j^*), e^{\varepsilon-2a}) \right|^2 e^{2\varepsilon a} dada'dxda'dxda' \]
\[ = \int_{\mathbb{R}} \int_{(-\varepsilon > \delta^3)} \int_{(x \times X)} \left| \hat{F}^{p_n}(a', x'; (-\varepsilon \sum_{j=1}^n x_j Y_j^*), e^{\varepsilon-2a}) \right|^2 e^{2\varepsilon a} dada'dxda'dxda' \]
\[ < \infty. \]

Therefore,
\[ \text{dis}((\pi_{\varepsilon}(F) - \sigma_{n, \delta}(F)), \mathcal{K}(L^2(\mathbb{R} \times X))) \leq \| \pi_{\varepsilon}(F) \circ (I - M_{\delta, 1}) - \sigma_{n, \delta}(F) \|_{H^{-S}} \to 0 \quad \text{as } \delta \to 0. \]

The Proposition follows, since $L^1_c$ is dense in $C^*(G_{n, \mu})$. \hfill \square
4.3. The two-dimensional orbits $\Omega_{\alpha}$ and the characters. The $C^*$-algebras of the groups $G_{V_k} = G_{n,\mu}/\mathbb{Z}$ have been determined as algebras of operator fields in [Lin-Lud]. We adapt this result to our present setting of almost $C_0(K)$-$C^*$-algebras.

**Definition 4.9.** For $a \in C^*(G_{n,\mu})$, let $\Phi(a)$ be the element of $C^*(\mathbb{R} \times V_0)$ defined by $\Phi(a)(\theta) := (\chi_0, a)$ for all $\theta \in \mathbb{R} \times V_0^*$. The mapping $\Phi : C^*(G_{n,\mu}) \to C^*(\mathbb{R} \times V_0)$ is a surjective homomorphism. Let the kernel of $\Phi$ be denoted by $I_K$, then $C^*(G_{n,\mu})/I_K \simeq C^*(\mathbb{R} \times V_0)$. For $\eta \in C_c(G_{n,\mu})$, the element $\Phi(\eta) \in C^*(\mathbb{R} \times V_0)$ is the continuous function with compact support given by

$$\Phi(\eta)(t,v_0) = \int_{V_0 \times \mathbb{R}} \eta(t,v_0,v,s)\,dvds \quad \text{for} \quad t \in \mathbb{R}, \ v_0 \in V_0.$$ 

Choose $\zeta \in C_c(V_1 \times \mathbb{R})$ with $\zeta \geq 0$ and $\int_{V_1 \times \mathbb{R}} \zeta(v,s)\,dvds = 1$, define the mapping $\beta : C_c(\mathbb{R} \times V_0) \to C_c(G_{n,\mu}) \subset C^*(G_{n,\mu})$ by

$$\beta(\varphi)(a,v_0,v,s) = \varphi(a,v_0)\zeta(v,s) \quad \text{for} \quad \varphi \in C_c(\mathbb{R} \times V_0), \ s \in \mathbb{R} \text{ and } v \in V_1.$$ 

It has been shown in [Lin-Lud] that $\beta$ can be extended to a linear mapping bounded by 1 from $C^*(\mathbb{R} \times V_0)$ into $C^*(G_{n,\mu})$, such that for every $\varphi \in C^*(\mathbb{R} \times V_0)$ we have $\Phi(\beta(\varphi)) = \varphi$.

**Definition 4.10.** Let $(\Omega_{f,k}) (f_k = (f_{k+}, f_{k-}) \in \mathcal{D}$ for all $k)$ be a properly converging sequence in $G_{n,\mu}$, whose limit set contains the orbits $\Omega_{f,0}$ and $\Omega_{f,\infty}$. Let $r_{k}, q_k \in \mathbb{R}$ be such that $|r_k \cdot f_{k+}| = 1$ and $|q_k \cdot f_{k-}| = 1$ for $k \in \mathbb{N}$. Then $\lim_{k \to \infty} r_k = -\infty$ and $\lim_{k \to \infty} q_k = +\infty$. Choose two positive sequences $(\rho_k)_k, (\kappa_k)_k$ such that $\kappa_k > q_k, -r_k < \rho_k$ for all $k \in \mathbb{N}$, $\lim_{k \to \infty} \kappa_k - q_k = \infty, \lim_{k \to \infty} \kappa_k - q_k = 0, \lim_{k \to \infty} \rho_k + r_k = \infty$ and $\lim_{k \to \infty} \frac{\kappa_k - q_k}{r_k} = 0, \lim_{k \to \infty} \frac{\rho_k + r_k}{q_k} = 0$. We say that the sequences $(\rho_k, \kappa_k)_k$ are adapted to the sequence $(f_k)_k$.

For $r \in \mathbb{R}$, let $U(r)$ be the unitary operator on $L^2(\mathbb{R})$ defined by

$$U(r)\xi(s) := \xi(s + r) \quad \text{for all} \xi \in L^2(\mathbb{R}) \text{ and } s \in \mathbb{R}.$$ 

**Definition 4.11.** Let $A = (A(f) \in \mathcal{B}, f \in \Gamma)$ be a field of bounded operators. We say that $A$ satisfies the generic condition if for every properly converging sequence $(\pi_{f_k})_k \subset G_{n,\mu}$ with $f_k \in \mathcal{D}$ for every $k \in \mathbb{N}$, which admits limit points $\pi_{f_{0,0}, f_{-}}, \pi_{f_{0,0}, f_{+}}$ and for every pair of sequences $(\rho_k, \kappa_k)_k$ adapted to the sequence $(f_k)_k$ we have that

\begin{align*}
(1) \quad & \lim_{k \to \infty} \|U(r_k) \circ A(f_k) \circ U(-r_k) \circ M_{\rho_k, +\infty} - A(f_0, f_+, 0) \circ M_{\rho_k, +\infty}\|_{op} = 0, \\
(2) \quad & \lim_{k \to \infty} \|U(q_k) \circ A(f_k) \circ U(-q_k) \circ M_{-\infty, \kappa_k} - A(f_0, f_-, 0) \circ M_{-\infty, \kappa_k}\|_{op} = 0.
\end{align*}

The following proposition had been proved in [Lin-Lud, Proposition 5.2].

**Proposition 4.12.** For every $a \in C^*(G_{n,\mu})$, the operator field $F(a)$ satisfies the generic condition.

We must show that on $\mathcal{D}$, our $C^*$-algebra satisfies the almost $C_0(K)$ conditions given in Definition 2.2. For $a \in C^*(G_{n,\mu})$ and $f = (f_0, f_+, f_-) \in V^*_{\text{gen}}$, we define the operator

$$\sigma_f(a) := \begin{aligned}
U(-r(f)) \circ \pi_{f_0, f_+, 0}(a) \circ U(r(f)) \circ M_{-\infty, \kappa(f) + r(f)} \\
+ U(-q(f)) \circ \pi_{f_0, f_-, 0}(a) \circ U(q(f)) \circ M_{q(f) - \rho(f), +\infty},
\end{aligned}$$

where

$$r(f) = -\ln(|f_+|), \quad q(f) = \ln(|f_-|), \quad \rho(f) = q(f) - r(f) = r(f)^{1/3}.$$ 

We have the following proposition.

**Proposition 4.13.** For all $f \in \mathcal{D}$, the operator field

$$f \mapsto \sigma_f(a) := \pi_f(a) - \sigma_f(a) \quad (a \in C^*(G_{n,\mu}))$$

is contained in $C_0(\mathcal{D}, K(L^2(\mathbb{R})))$. 

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Proof. Let $a \in C^*(G_{n,\mu})$. We know that $\pi_f(a)$ is a compact operator for any $f \in V_{gen}^*$, that the mapping $f \mapsto \pi_f(a)$ is norm continuous and that $\lim_{f \to \infty} \pi_f(a) = 0$ by Corollary 3.2 and Proposition 4.2 in [Lin-Lud]. If $F \in L^1_c$, then the kernel function $F_{f_0,f_1}$ of the operator $\pi_{(f_0,f_1,0)} \circ M_{[\rho(f),\infty]}$ is given by

$$F_{f_0,f_1}(s,t) = \int_{\mathbb{R}^2} \delta(s-t,t \cdot f_1) \chi_{[\rho(f),\infty]}(t) \, dt.$$  

The function $F_{f_0,f_1}$ is of compact support and $\rho$ is continuous. Hence the mapping $f \mapsto \pi_{(f_0,f_1,0)} \circ M_{[\rho(f),\infty]}$ is norm continuous on $D$ and for every $f \in D$, the operator $\pi_{(f_0,f_1,0)} \circ M_{[\rho(f),\infty]}$ is compact. Since

$$\rho(f) = \ln(|f_-|)^{1/3} + \ln(|f_+|)$$

we go to infinity as $\|f\|$ goes to infinity, it follows that $\pi_{(f_0,f_1,0)} \circ M_{[\rho(f),\infty]} = 0$ if $\|f\|$ is big enough. Similar properties hold for the mapping $f \mapsto \pi_{(f_0,f_1,0)} \circ M_{-\infty,\rho(f)}$ on $D$.

Since the boundary $\partial D$ of $D$ is the set $S \cup \mathbb{R}$, the generic condition tells us that

$$\lim_{f \to \partial D} \|\sigma_D(f)(a)\| = 0.$$ 

Hence the mapping $f \mapsto \sigma_D(f)(F)$ is contained in $C_0(\overline{D} \cap K(L^2(\mathbb{R})))$. The proposition follows from the density of $L^2_c$ in $C^*(G_{n,\mu})$. □

4.4. The C*-algebras of the groups $G_{n,\mu}$. Let $\Gamma_i \subseteq g_{n,\mu}/G_{n,\mu}$ be given as in Section 3.5 and $\Gamma = \sqcup \Gamma_i$.

Definition 4.14. (1) For $f \in D$ and $\phi \in l^\infty(\Gamma)$, let

$$\sigma_f(\phi) := U(-r(f)) \circ \phi(f_0,f_1,0) \circ U(r(f)) \circ M_{-\infty,\rho(f)+r(f)}$$

$$+ U(-q(f)) \circ \phi(f_0,0,f_-) \circ U(q(f)) \circ M_{\rho(f)-\rho(f)+\infty}.$$ 

(2) Let $\varphi = (\varphi(f) \in \mathcal{B}, f \in \Gamma)$ be a field of bounded operators such that the restriction of the field $\varphi$ to the set of characters $\Gamma_0$ is contained in $C_0(\Gamma_0)$. We get the element $\varphi(0) \in C^*(\mathbb{R} \times V_0)$ determined as in Definition 4.9 by the condition $\gamma(\varphi(0)) = \varphi(\gamma)$ for $\gamma \in \Gamma_0$. We can then define as in Definition 4.9 that

$$\sigma_f(\varphi) := \beta(\varphi(0)) \in \mathcal{B}(L^2(\mathbb{R}))$$

for $f \in S$.

Definition 4.15. Let $D^*(G_{n,\mu})$ be the subset of $l^\infty(G_{n,\mu})$ defined as a set of all the operator fields $\phi$ defined over $G_{n,\mu}$ such that the mappings $\gamma \mapsto \phi(\gamma)$ are norm continuous and vanish at infinity on the sets $\Gamma_0$ and $\Gamma_2$ and such that $\phi(f) \in K(L^2(\mathbb{R}))$ for all $f \in D$. Moreover, each $\phi$ must fulfill the following conditions:

1. For $\varepsilon \in \{+,-\}$, we have

$$\lim_{\delta \to 0} \text{dis}(\phi(\varepsilon) - \sigma_{n,\delta}(\phi), K(L^2(\mathbb{R} \times X))) = 0,$$

and

$$\lim_{\delta \to 0} \text{dis}(\phi^*(\varepsilon) - \sigma_{n,\delta}(\phi^*), K(L^2(\mathbb{R} \times X))) = 0.$$ 

2. The mappings $D \ni f \mapsto (\phi(f) - \sigma_f(\phi))$ and $D \ni f \mapsto (\phi(f)^* - \sigma_f(\phi^*))$ are contained in $C_0(D, K(L^2(\mathbb{R})))$.

3. The mappings $S \ni f \mapsto (\phi(f) - \sigma_f(\phi))$ and $S \ni f \mapsto (\phi(f)^* - \sigma_f(\phi^*))$ are contained in $C_0(S, K(L^2(\mathbb{R})))$.

Theorem 4.16. The C*-algebra of $G_{n,\mu}$ is an almost $C_0(K)$-C*-algebra. In particular, the Fourier transform maps $C^*(G_{n,\mu})$ onto the subalgebra $D^*(G_{n,\mu})$ of $l^\infty(\Gamma)$.

Proof. Propositions 4.8 and 4.13 show that the Fourier transform maps $C^*(G_{n,\mu})$ into $D^*(G_{n,\mu})$. The conditions on $D^*(G_{n,\mu})$ imply that $D^*(G_{n,\mu})$ is a closed involutive subspace of $l^\infty(\Gamma)$. It follows from [ILL] that $D^*(G_{n,\mu})$ is a C*-subalgebra of $l^\infty(\Gamma)$ and that $\mathcal{F}_{n,\mu}(C^*(G_{n,\mu})) = D^*(G_{n,\mu})$. □
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