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Abstract

We construct an infinite dimensional non-unital Banach algebra $A$ and $a \in A$ such that the sets $\{za^n : z \in \mathbb{C}, n \in \mathbb{N}\}$ and $\{(1+a)^n a : n \in \mathbb{N}\}$ are both dense in $A$, where $1$ is the unity in the unitalization $A# = A \oplus \text{span}\{1\}$ of $A$. As a byproduct, we get a hypercyclic operator $T$ on a Banach space such that $T \oplus T$ is non-cyclic and $\sigma(T) = \{1\}$.

MSC: 47A16, 46J45

Keywords: Hypercyclic operators; supercyclic operators; Banach algebras

1 Introduction

All vector spaces in this article are over the field $\mathbb{C}$ of complex numbers. As usual, $\mathbb{R}$ is the field of real numbers, $T = \{x \in \mathbb{C} : |x| = 1\}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$, $\mathbb{R}_+ = [0, \infty)$, $\mathbb{N}$ is the set of positive integers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. If $X$ and $Y$ are topological vector spaces, $L(X,Y)$ stands for the space of continuous linear operators from $X$ to $Y$. We write $L(X)$ instead of $L(X,X)$ and $X^*$ instead of $L(X,\mathbb{C})$. For $T \in L(X,Y)$, the dual operator $T^* \in L(Y^*,X^*)$ is defined as usual: $T^* f = f \circ T$. Recall that $T \in L(X)$ is called hypercyclic (respectively, supercyclic) if there is $x \in X$ such that the orbit $O(T,x) = \{T^n x : n \in \mathbb{Z}_+\}$ (respectively, the projective orbit $\{zT^n x : z \in \mathbb{C}, n \in \mathbb{Z}_+\}$) is dense in $X$. Such an $x$ is called a hypercyclic vector (respectively, a supercyclic vector) for $T$. We refer to [1] and references therein for additional information on hypercyclicity and supercyclicity. Recall that a function $\pi : A \rightarrow \mathbb{R}_+$ defined on a complex algebra $A$ is called submultiplicative if $\pi(ab) \leq \pi(a)\pi(b)$ for any $a,b \in A$. A Banach algebra is a complex (maybe non-unital) algebra $A$ with a complete submultiplicative norm (if $A$ is unital, it is usually also assumed that $\|1\| = 1$, where $1$ is the unity in $A$). We say that $A$ is non-trivial if $A \neq \{0\}$.

Definition 1.1. Let $A$ be a Banach algebra. We say that $A$ is supercyclic if there is $a \in A$ for which $\{za^n : z \in \mathbb{C}, n \in \mathbb{N}\}$ is dense in $A$. Such an $a$ is called a supercyclic element of $A$. We say that $A$ is almost hypercyclic if there is $a \in A$ for which $\{(1+a)^n a : n \in \mathbb{N}\}$ is dense in $A$. Such an $a$ is called an almost hypercyclic element of $A$. Finally, we say that a Banach algebra $A$ is chaotic if there is $a \in A$ which is a supercyclic and an almost hypercyclic element of $A$. In other words, both $\{za^n : z \in \mathbb{C}, n \in \mathbb{N}\}$ and $\{(1+a)^n a : n \in \mathbb{N}\}$ are dense in $A$. Such an $a$ is called a chaotic element of $A$.

In the above definition, $1$ is the unit element in the unitalization $A# = A \oplus \text{span}\{1\}$ of $A$. Note that $a$ is a supercyclic element of $A$ if and only if $a$ is a supercyclic vector for the multiplication operator

$$M_a \in L(A), \quad M_a b = ab$$

(1.1)

and $a$ is an almost hypercyclic element of $A$ if and only if $a$ is a hypercyclic vector for $I + M_a$. There is no point to consider 'hypercyclic Banach algebras' in the obvious sense. Indeed, in [10] it is observed that a multiplication operator on a commutative Banach algebra is never hypercyclic. Obviously, supercyclic as well as almost hypercyclic Banach algebras are commutative and separable.

Theorem 1.2. There exists a chaotic infinite dimensional Banach algebra $A$.

In order to emphasize the value of Theorem 1.2, we would like to mention few related facts. A Banach algebra is called radical if it coincides with its Jacobson radical [4]. If $A$ is a Banach algebra and $X$ is a Banach $A$-bimodule [4], then $D \in L(A, X)$ is called a derivation if $D(ab) = (Da)b + a(Db)$ for each $a, b \in A$. 

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A Banach algebra $A$ is called weakly amenable if every derivation $D : A \to A^*$ (with the natural bimodule structure on $A^*$) has the shape $Da = ax - xa$ for some $x \in A^*$. It is well-known [4] that a commutative Banach algebra $A$ is weakly amenable if and only if there is no non-zero derivations $D : A \to X$ taking values in a commutative Banach $A$-bimodule $X$.

**Theorem 1.3.** Let $A$ be a supercyclic Banach algebra of dimension $> 1$. Then $A$ is infinite dimensional, radical and weakly amenable.

According to Theorem 1.3, Theorem 1.2 provides an infinite dimensional radical weakly amenable Banach algebra. We would like to mention the work [7] by Loy, Read, Runde, and Willis, who constructed a non-unital Banach algebra, generated by one element $x$ of radius less than one. Hence, it is automatically radical and weakly amenable. Theorem 1.3 shows that the same properties are forced by supercyclicity. It is also worth mentioning that Read [8] constructed a commutative amenable radical Banach algebra, but this algebra is not generated by one element.

**Proposition 1.4.** Let $A$ be a non-trivial commutative Banach algebra and $M = cI + M_a \in L(A)$, where $a \in A$ and $c \in \mathbb{C}$. Then $M \oplus M$ is non-cyclic.

**Proof.** Let $(x, y) \in A^2$. If $M_x = M_y = 0$, then $(M \oplus M)^n(x, y) = c^n(x, y)$ for every $n \in \mathbb{Z}_+$ and therefore $(x, y)$ is not a cyclic vector for $M_a \oplus M_a$. Otherwise, the operator $T \in L(A^2, A)$, $T(u, v) = yu - xv$ is non-zero. Moreover, $T((M \oplus M)^n(x, y)) = T((cI + a)^n x, (cI + a)^n y) = y(cI + a)^n x - x(cI + a)^n y = 0$ since $A$ is commutative. Thus $(M \oplus M)^n(x, y) \in \ker T$ for each $n \in \mathbb{Z}_+$. Since $\ker T$ is a proper closed linear subspace of $A^2$, $(x, y)$ again is not a cyclic vector for $M \oplus M$. □

By Proposition 1.4, Theorem 1.2 provides hypercyclic operators $T$ with non-cyclic $T \oplus T$. The existence of such operators was used to be an open problem until De La Rosa and Read [5] (see also [2] and [1]) constructed such operators. One can observe that the spectra of the operators in [5, 2] contain a disk centered at 0 of radius $> 1$. On the other hand [1], any separable infinite dimensional complex Banach space supports hypercyclic operators with the spectrum being the singleton $\{1\}$. It remained unclear whether a hypercyclic operator $T$ with non-cyclic $T \oplus T$ can have small spectrum. Theorem 1.2 provides such an operator. Indeed, by Theorem 1.2, there are an infinite dimensional Banach algebra $A$ and $a \in A$ such that $T = I + M_a$ is hypercyclic. By Theorem 1.3, $A$ is radical and therefore $M_a$ is quasinilpotent. Hence the spectrum $\sigma(T)$ of $T$ is $\{1\}$. Thus we arrive to the following corollary.

**Corollary 1.5.** There exists a hypercyclic continuous linear operator $T$ on an infinite dimensional Banach space such that $T \oplus T$ is non-cyclic and $\sigma(T) = \{1\}$.

It seems to be of independent interest that supercyclic operators $T$ with non-cyclic $T \oplus T$ can be found among multiplication operators on commutative Banach algebras, while hypercyclic operators $T$ with non-cyclic $T \oplus T$ can be of the shape identity plus a multiplication operator.

## 2 Proof of Theorem 1.3

Since a Banach space of finite dimension $> 1$ supports no supercyclic operators (see [12]), a supercyclic Banach algebra of dimension $> 1$ must be infinite dimensional. According to [10, Proposition 3.4], an infinite dimensional commutative Banach algebra $B$ is radical if there is $b \in B$ for which the multiplication operator $Mb$ is supercyclic. Since a supercyclic Banach algebra of dimension $> 1$ is infinite dimensional, commutative and has a supercyclic multiplication operator, $A$ is radical.

It remains to show that that $A$ is weakly amenable. Assume the contrary. Then there is a commutative Banach $A$-bimodule $X$ and a non-zero derivation $D \in L(A, X)$. Since $A$ is supercyclic, there is $a \in A$ such that $\{za^n : z \in \mathbb{C}, n \in \mathbb{N}\}$ is dense in $A$. Since $\dim A > 1$, $\Omega_m = \{za^n : z \in \mathbb{C}, n \geq m\}$ is dense in $A$ for each $m \in \mathbb{N}$. Consider the operator $M \in L(A, X)$, $Mb = bDa$. Since $X$ is commutative and $D$ is a derivation, we have $D(a^n) = na^{n-1}Da$ for $n \geq 2$. If $M = 0$, then $D(a^n) = na^{n-1}Da = nM(a^{n-1}) = 0$ for $n \geq 2$. Hence $D$ vanishes on the dense set $\Omega_2$. Since $D$ is continuous, $D = 0$, which is a contradiction.
Hence \( M \neq 0 \) and therefore \( M^* \neq 0 \). Thus there is \( f \in X^* \) such that \( g = M^* f \) is a non-zero element of \( A^* \). Then for each \( n \in \mathbb{N} \), we have \( g(a^n) = M^* f(a^n) = f(a^n Da) = \frac{f(D(a^{n+1}))}{n+1} \). Hence

\[
|g(a^n)| = \left| \frac{f(D(a^{n+1}))}{n+1} \right| \leq C \frac{\|a^n\|}{n+1}, \quad \text{where } C = \|D\| \|f\| \|a\|.
\]

Now let \( m \in \mathbb{N} \) be such that \( \frac{C}{m+1} < \frac{|g|}{2} \) and \( W = \{ u \in A : |g(u)| > \frac{|g|}{2} \} \). Clearly \( W \) is non-empty and open. By the last display, \( \Omega_m \cap W = \emptyset \), which contradicts the density of \( \Omega_m \) in \( A \). This contradiction completes the proof of Theorem 1.3.

### 3 Proof of Theorem 1.2

From now on, \( \mathbb{P} \) is the algebra \( \mathbb{C}[z] \) of polynomials with complex coefficients in one variable \( z \). Clearly, \( \mathbb{P}_0 = \{ p \in \mathbb{P} : p(0) = 0 \} \) is an ideal in \( \mathbb{P} \) of codimension 1. There is a sequence \( \{ p_n \}_{n \in \mathbb{N}} \) in \( \mathbb{P}_0 \) such that

\[
\{ p_n : n \in \mathbb{N} \} \text{ is dense in } \mathbb{P}_0 \text{ with respect to any seminorm on } \mathbb{P}_0.
\]

Indeed, (3.1) is satisfied if, for instance, \( \{ p_n : n \in \mathbb{N} \} \) is the set of all polynomials in \( \mathbb{P}_0 \) with coefficients from a fixed dense countable subset of \( \mathbb{C} \), containing 0.

**Lemma 3.1.** Let \( \pi \) be a non-zero submultiplicative seminorm on \( \mathbb{P}_0 \) and \( \{ p_k \}_{k \in \mathbb{N}} \) is a sequence in \( \mathbb{P}_0 \) satisfying (3.1). Assume also that there exist sequences \( \{ n_k \}_{k \in \mathbb{N}} \) and \( \{ m_k \}_{k \in \mathbb{N}} \) of positive integers and a sequence \( \{ c_k \}_{k \in \mathbb{N}} \) of complex numbers such that \( \pi(c_k z^{n_k} - p_k) \to 0 \) and \( \pi(z(1 + z)^{m_k} - p_k) \to 0 \). Then \( \pi \) is a norm and the completion \( A \) of \( (\mathbb{P}_0, \pi) \) is an infinite dimensional chaotic Banach algebra with \( z \) as a chaotic element.

**Proof.** Let \( I = \{ q \in \mathbb{P}_0 : \pi(q) = 0 \} \). Since \( \pi \) is submultiplicative, \( I \) is an ideal in \( \mathbb{P}_0 \) and therefore in \( \mathbb{P} \). Since \( \pi \) is non-zero, \( I \neq \mathbb{P}_0 \). Thus \( \mathbb{P}_0/I \) with the norm \( \|q + I\| = \pi(q) \) is a non-trivial complex algebra with a submultiplicative norm. Since \( \pi(z(1 + z)^{m_k} - p_k) \to 0 \), (3.1) implies that the operator \( p + I \mapsto (1 + z)p + I \) on \( \mathbb{P}_0/I \) is hypercyclic with the hypercyclic vector \( z + I \). Since there is no hypercyclic operator on a non-trivial finite dimensional normed space \([12]\), \( \mathbb{P}_0/I \) is infinite dimensional and therefore \( I \) has infinite codimension in \( \mathbb{P} \). Since the only ideal in \( \mathbb{P} \) of infinite codimension is \( \{ 0 \} \), \( I = \{ 0 \} \) and therefore \( \pi \) is a norm.

Thus the completion \( A \) of \( (\mathbb{P}_0, \pi) \) is an infinite dimensional Banach algebra. Conditions \( \pi(c_k z^{n_k} - p_k) \to 0 \) and \( \pi(z(1 + z)^{m_k} - p_k) \to 0 \) together with (3.1) imply that \( A \) is chaotic with \( z \) as a chaotic element. \( \square \)

It remains to construct a seminorm on \( \mathbb{P}_0 \), which will allow us to apply Lemma 3.1.

#### 3.1 Ideals in \( A[k] \) and submultiplicative norms on \( \mathbb{P} \)

For \( k \in \mathbb{N} \), we consider the commutative Banach algebra \( A[k] \) of the power series

\[
a = \sum_{n \in \mathbb{Z}_+^k} a_n u_1^{n_1} \ldots u_k^{n_k}, \quad \text{with } \|a\|_{[k]} = \sum_{n \in \mathbb{Z}_+^k} |a_n| < \infty
\]

with the natural multiplication. We will treat the elements of \( A[k] \) both as power series and as continuous functions \( u \mapsto a(u_1, \ldots, u_k) \) on \( \mathbb{D}^k \), holomorphic on \( \mathbb{D}^k \). Note that as a Banach space \( A[k] \) is \( \ell_1(\mathbb{Z}_+^k) \). In particular, the underlying Banach space of \( A[k] \) can be treated as the dual space of \( c_0(\mathbb{Z}_+^k) \), which allows us to speak about the weak-* topology on \( A[k] \).

For a non-empty open subset \( U \) of \( \mathbb{C} \) we also consider the complex algebra \( \mathcal{H}_U \) of holomorphic functions \( f : U \to \mathbb{C} \) endowed with the Fréchet space topology of uniform convergence on compact subsets of \( U \). For \( \gamma > 0 \), we write \( \mathcal{H}_\gamma \) instead of \( \mathcal{H}_{\gamma \mathbb{D}} \).

If \( \xi = (\xi_1, \ldots, \xi_k) \in \mathbb{P}_0^k \) and \( a \in A[k] \), we can consider \( a(\xi_1, \ldots, \xi_k) \) as a power series

\[
a(\xi_1, \ldots, \xi_k)(z) = a(\xi_1(z), \ldots, \xi_k(z)) = \sum_{m=1}^{\infty} a_m(a, \xi) z^m,
\]

(3.2)
which converges uniformly on the compact subsets of the disk $\gamma(\xi)\mathbb{D}$, where

$$ \gamma(\xi) = \sup\{c > 0 : \xi_j(c\mathbb{D}) \subseteq \mathbb{D} \text{ for } 1 \leq j \leq k\} > 0. $$

By the Hadamard formula, $\lim_{m \to \infty} |\alpha_m(a, \xi)|^{1/m} \leq \frac{1}{\gamma(\xi)}$ for each $a \in A^k$. By the uniform boundedness principle, $\lim_{m \to \infty} \|\alpha_m(\cdot, \xi)\|^{1/m} \leq \frac{1}{\gamma(\xi)}$, where the norm is taken in $(A^k)^*$. Hence the map

$$ \Phi_\xi : A^k \to \mathcal{H}_{\gamma(\xi)}, \quad \Phi_\xi(a) = a(\xi_1, \ldots, \xi_k) $$

is a continuous algebra homomorphism from the Banach algebra $A^k$ to the Fréchet algebra $\mathcal{H}_{\gamma(\xi)}$ of holomorphic complex valued functions on the disk $\gamma(\xi)\mathbb{D}$.

**Remark 3.2.** Note that if $U$ is a connected non-empty open subset of $\mathbb{C}$ and all zeros of a polynomial $p \in \mathbb{P}$ of degree $n \in \mathbb{N}$ are in $U$, then the ideal $J_p$, generated by $p$ in the algebra $\mathcal{H}_U$, is closed and has codimension $n$. It consists of all $f \in \mathcal{H}_U$ such that every zero of $p$ of order $k \in \mathbb{N}$ is also a zero of $f$ of order $\geq k$. We write $p|f$ to denote the inclusion $f \in J_p$. Note that $\mathcal{H}_U = J_p \oplus \text{span}\{1, z, \ldots, z^{n-1}\}$.

We use the following notation. If $\xi \in \mathbb{P}_0^k$ and $q \in \mathbb{P}$ has all its zeros in the disk $\gamma(\xi)\mathbb{D}$, then

$$ I_{\xi,q} = \{a \in A^k : q|\Phi_\xi(a)\} $$

with $\Phi_\xi(a)$ considered as an element of $\mathcal{H}_{\gamma(\xi)}$. In the case $q = z^n$ with $n \in \mathbb{N}$, we have

$$ I_{\xi,z^n} = \{a \in A^k : \alpha_j(a, \xi) = 0 \text{ for } 0 \leq j < n\}, $$

where $\alpha_j(a, \xi)$ are defined in (3.2). Finally,

$$ I_\xi = \ker \Phi_\xi = \bigcap_{n=1}^\infty I_{\xi,z^n}. $$

**Lemma 3.3.** Let $\xi = (\xi_1, \ldots, \xi_k) \in \mathbb{P}_0^k$ be such that $\xi_1 = z$. Then $I_\xi$ is a closed ideal in $A^k$ and for each $q \in \mathbb{P}$, whose zeros are in the disk $\gamma(\xi)\mathbb{D}$, $I_{\xi,q}$ is closed ideal in $A^k$ of codimension $\deg q$. Moreover, $I_\xi \subseteq I_{\xi,q}$ and

$$ \|a + I_{\xi,z^n}\|_{A^k/I_{\xi,z^n}} \to \|a + I_\xi\|_{A^k/I_\xi} \quad \text{as } n \to \infty \text{ for each } a \in A^k. $$

Furthermore, if $q_n \in \mathbb{P}$ for $n \in \mathbb{N} \cup \{\infty\}$ are polynomials of degree $m \in \mathbb{N}$, whose zeros are in $\gamma(\xi)\mathbb{D}$ and the sequence $\{q_n\}_{n \in \mathbb{N}}$ converges to $q_\infty$ as $n \to \infty$ (in the usual sense in the finite dimensional space of polynomials of degree $\leq m$), then

$$ \|a + I_{\xi,q_n}\|_{A^k/I_{\xi,q_n}} \to \|a + I_\xi,q_\infty\|_{A^k/I_\xi,q_\infty} \quad \text{as } n \to \infty \text{ for each } a \in A^k. $$

**Proof.** For brevity, we denote $\gamma = \gamma(\xi)$. It is straightforward to verify that $\Phi_\xi : A^k \to \mathcal{H}_\xi$ is not just continuous but also weak-* continuous. That is $\Phi_\xi$ is continuous when $A^k$ is equipped with the weak-* topology and the Fréchet space $\mathcal{H}_\xi$ carries its weak (=weak-* topology.

Since $I_{\xi,q} = \Phi_\xi^{-1}(J_q)$ for every polynomial $q$ whose zeros are all in $\gamma\mathbb{D}$ and $I_\xi = \Phi_\xi^{-1}(0)$, we see that the ideals $I_{\xi,q}$ and $I_\xi$ are weak-* closed and therefore closed in $A^k$. Using the equality $\xi_1 = z$, one can readily see that $I_{\xi,q} \oplus \text{span}\{1, u_1, \ldots, u_r\} = A^k$, where $r = \deg q - 1$. Thus $I_{\xi,q}$ has codimension $\deg q$ in $A^k$.

Obviously, $I_\xi \subseteq I_{\xi,z^{n+1}} \subseteq I_{\xi,z^n}$ for every $n \in \mathbb{N}$. Therefore, the sequence $\|a + I_{\xi,z^n}\|_{A^k/I_{\xi,z^n}}$ is increasing and bounded above by $\|a + I_\xi\|_{A^k/I_\xi}$ for every $a \in A^k$. Hence

$$ c = \lim_{n \to \infty} \|a + I_{\xi,z^n}\|_{A^k/I_{\xi,z^n}} \leq \|a + I_\xi\|_{A^k/I_\xi} = c_1. $$

The proof of (3.6) will be complete if we show that $c_1 \leq c$. 


By definition of the quotient norms, we can find $b_n \in I_{\xi, z_n}$ such that $\|a + b_n\|_k \to c$. By the Banach–Alaoglu theorem, the bounded sequence $\{b_n\}$ has a weak−$*$ accumulation point $b$ in $A[k]$. Since $b_m \in T_{\xi, z_n}$ for $m \geq n$ and each $I_{\xi, z_n}$ is weak−$*$ closed, $b$ belongs to every $I_{\xi, z_n}$ and therefore to their intersection $I_\xi$: $b \in I_\xi$. Since the norm is weak−$*$ upper semicontinuous (a straightforward consequence of the Hahn–Banach theorem) and $\|a + b_n\|_k \to c$, we have $\|a + b\|_k \leq c$. Since $b \in I_\xi$, $c_1 = \|a + I_\xi\|_{A[k]/I_\xi} \leq \|a + b\| \leq c$, which completes the proof of (3.6).

It remains to prove (3.7). Let $m \in \mathbb{N}$ and $\mathcal{P}_m$ be the $(m + 1)$-dimensional space of polynomials of degree $\leq m$. Let also $q_n \in \mathcal{P}_m$ for $n \in \mathbb{N} \cup \{\infty\}$ be polynomials of degree exactly $m$, whose zeros are all in $\gamma \mathbb{D}$ and the sequence $\{q_n\}_{n \in \mathbb{N}}$ converges to $q_{\infty}$ as $n \to \infty$ in the finite dimensional space $\mathcal{P}_m$. Since $\xi_1 = z \in \mathbb{P} \subset \Phi_{\xi}(A[k])$. Indeed, $\Phi_{\xi}(a) = p$ if $p \in \mathbb{P}$ and $a(u_1, \ldots, u_k) = p(u_1)$. Furthermore, $\Phi_{\xi}^{-1}(\mathbb{P})$ contains the unital subalgebra generated by $u_1, \ldots, u_k$ and therefore is dense in $A[k]$. It is an easy exercise that in every topological vector space $X$ the intersection $L \cap M$ of a dense in $X$ linear subspace $L$ and a finite codimensional closed subspace $M$ is dense in $M$. It follows that

$$\Phi_{\xi}^{-1}(\mathbb{P} \cap J_q) = \Phi_{\xi}^{-1}(\mathbb{P}) \cap I_{\xi,q}$$

is dense in $I_{\xi,q}$

for every polynomial $q$ whose zeros are all in $\gamma \mathbb{D}$.

Now take $a \in \Phi_{\xi}^{-1}(\mathbb{P})$ and denote $c_n = \|a + I_{\xi,q_n}\|_{A[k]/I_{\xi,q_n}}$ for $n \in \mathbb{N} \cup \{\infty\}$. By the above display, for each $\varepsilon > 0$, we can pick $b \in \Phi_{\xi}^{-1}(\mathbb{P} \cap J_{q_{\infty}})$ such that $\|a + b\|_k \leq c_{\infty} + \varepsilon$. Since $b \in \Phi_{\xi}^{-1}(\mathbb{P} \cap J_{q_{\infty}})$, $\Phi_{\xi}(b) = pq_{\infty}$ for some $p \in \mathbb{P}$. Since $p\mathcal{P}_m$ is an $(m + 1)$-dimensional subspace of $\mathbb{P} \subset \Phi_{\xi}(A[k])$, we can find an $(m + 1)$-dimensional subspace $L$ of $A[k]$ such that $R = \Phi_{\xi}|_L : L \to p\mathcal{P}_m$ is a linear isomorphism. Set $b_n = b + R^{-1}(p(q_n - q_{\infty}))$. Since every linear operator on a finite dimensional topological vector space is continuous and $pq_{\infty} \to pq_{\infty}$, we have $\|b_n - b\|_k = 0$. On the other hand, the construction of $b_n$ yields $\Phi_{\xi}(b_n) = pq_n \in J_{q_n}$. Hence $b_n \in I_{\xi,q_n}$ and therefore

$$c_n = \|a + I_{\xi,q_n}\|_{A[k]/I_{\xi,q_n}} \leq \|a + b_n \|_k \to \|a + b\|_k \leq c_{\infty} + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $\lim_{n \to \infty} c_n \leq c_{\infty}$. In order to verify (3.7) for $a$ it now suffices to show that $\lim_{n \to \infty} c_n \geq c_{\infty}$. Pick $d_n \in I_{\xi,q_n}$ such that $\|a + d_n\|_k - c_n \to 0$. Clearly, $\{d_n\}$ is bounded. By the Banach–Alaoglu theorem, every bounded sequence in $A[k]$ has a weak−$*$ convergent subsequence. Thus, if $\lim_{n \to \infty} c_n \geq c_{\infty}$ fails, passing to a subsequence, if necessary, we can assume that $c_n \to r < c_{\infty}$ and $\{d_n\}$ is weak−$*$ convergent to $d \in A[k]$. Since $d_n \in I_{\xi,q_n}$, $\Phi_{\xi}(d_n) = q_n f_n$ with $f_n \in \mathcal{H}_\gamma$ for every $n \in \mathbb{N}$. Since $\{d_n\}$ is weak−$*$ convergent to $d$ and $\Phi_{\xi}$ is weak−$*$ continuous, $\{q_n f_n\}$ converges weakly in $\mathcal{H}_\gamma$ to $\Phi_{\xi}(d) \in \mathcal{H}_\gamma$. Since $\mathcal{H}_\gamma$ is a nuclear (and therefore Montel) Fréchet space, weak and strong convergence of sequences in $\mathcal{H}_\gamma$ coincide. Hence $q_n f_n \to \Phi_{\xi}(d)$ in the Fréchet space $\mathcal{H}_\gamma$. That is the sequence $\{q_n f_n\}$ of holomorphic on $\gamma \mathbb{D}$ functions converges to $\Phi_{\xi}(d)$ uniformly on compact subsets of $\gamma \mathbb{D}$. Since $q_n \to q_{\infty}$ and all zeros of $q_{\infty}$ are in $\gamma \mathbb{D}$, there exists $\gamma' < \gamma$ and $\delta > 0$ such that $\|q_n(z)\| \geq \delta$ whenever $\gamma' < |z| < \gamma$. It follows that $\{f_n\}$ converges uniformly on compact sets of the last annulus. Since each $f_n$ is holomorphic on $\gamma \mathbb{D}$, $\{f_n\}$ converges in $\mathcal{H}_\gamma$ to some $f_{\infty} \in \mathcal{H}_\gamma$. Since $q_n \to q_{\infty}$, we have $q_n f_n \to q_{\infty} f_{\infty}$ in $\mathcal{H}_\gamma$. Thus $\Phi_{\xi}(d) = q_{\infty} f_{\infty} \in J_{q_{\infty}}$ and therefore $d \in I_{\xi,q_{\infty}}$. Using the weak−$*$ upper semicontinuity of the norm, we obtain

$$c_{\infty} \leq \|a + d\|_k \leq \lim_{n \to \infty} \|a + d_n\| = \lim_{n \to \infty} c_n = r < c_{\infty}.$$

This contradiction completes the proof of (3.7) for $a$ form the dense subset $\Phi_{\xi}^{-1}(\mathbb{P})$ of $A[k]$. Note also that if a locally uniformly continuous and locally uniformly bounded sequence of maps between two complete metric spaces converges to a continuous map pointwise on a dense set, then it converges everywhere. It remains to notice that each seminorm $a \mapsto \|a + I_{\xi,q_n}\|_{A[k]/I_{\xi,q_n}}$ is Lipschitz with constant 1 and is bounded above by the norm $\| \cdot \|_k$. Thus the previous remark allows to extend the validity of (3.7) for $a \in \Phi_{\xi}^{-1}(\mathbb{P})$ to its validity for each $a \in A[k]$. This completes the proof of (3.7) and of the lemma. \qed
As we have already mentioned, \( P \subseteq \Phi_\xi(A^{[k]}) \) if \( \xi \in P^0_0 \) and \( \xi_1 = z \). Hence we can use the above ideals to define seminorms on \( P \). Since \( I_\xi = \ker \Phi_\xi \) and \( \Phi_\xi(A^{[k]}) \supseteq P \), we can write

\[
\pi_\xi : P \to \mathbb{R}^+, \quad \pi_\xi(p) = \|\Phi_\xi^{-1}(p)\|_{A^{[k]}/I_\xi} = \inf\{\|a\|_{[k]} : a \in A^{[k]}, \Phi_\xi(a) = p\}.
\] (3.8)

By Lemma 3.3, \( I_\xi \) is a closed ideal in \( A^{[k]} \) and therefore \( \pi_\xi \) is a submultiplicative norm on \( P \).

If additionally \( q \in P \) has all its zeros in the disk \( \gamma(\xi)\mathbb{D} \), then using the closeness of the ideal \( I_{\xi,q} \) in \( A^{[k]} \) and the inclusion \( I_\xi \subset I_{\xi,q} \), we can define

\[
\pi_{\xi,q} : P \to \mathbb{R}^+, \quad \pi_{\xi,q}(p) = \|\Phi_\xi^{-1}(p) + I_{\xi,q}\|_{A^{[k]}/I_{\xi,q}} = \inf\{\|a\|_{[k]} : a \in A^{[k]}, q|(p - \Phi_\xi(a))\}.
\] (3.9)

The function \( \pi_{\xi,q} \) is a submultiplicative seminorm on \( P \).

**Lemma 3.4.** Let \( k \in \mathbb{N}, \xi' = (\xi_1, \ldots, \xi_{k+1}) \in P_0^{k+1} \) with \( \xi_1 = z = (\xi_1, \ldots, \xi_k) \). Then \( \pi_{\xi'}(p) \leq \pi_\xi(p) \) for all \( p \in P \). Moreover, if \( U \) is a connected open subset of \( \gamma(\xi)\mathbb{D} \), \( 0 \in U, \xi_{k+1}(U) \subseteq \mathbb{D} \) and \( q \in P \setminus \{0\} \) is a divisor of \( \xi_{k+1} \) and has all its zeros in \( U \), then \( \pi_{\xi,q}(p) \leq \pi_{\xi'}(p) \) for every \( p \in P \).

**Proof.** For any \( p \in P \) and \( a \in A^{[k]} \) satisfying \( \Phi_\xi(a) = p \), we have \( \Phi_{\xi'}(b) = p \) and \( \|a\|_{[k]} = \|b\|_{[k+1]} \) with \( b(u_1, \ldots, u_{k+1}) = a(u_1, \ldots, u_k) \). By (3.8), \( \pi_{\xi'}(p) \leq \pi_\xi(p) \) for each \( p \in P \). Now assume that \( U \) is a connected open subset of \( \gamma(\xi)\mathbb{D} \), \( 0 \in U, \xi_{k+1}(U) \subseteq \mathbb{D} \) and \( q \in P \setminus \{0\} \) is a divisor of \( \xi_{k+1} \) and has all its zeros in \( U \). Let \( p \in P \) and \( a \in A^{[k+1]} \) be such that \( \Phi_{\xi'}(a) = p \). By definition of \( A^{[k+1]} \),

\[
a = b_0 + \sum_{n=1}^\infty b_n u_{k+1}^n, \quad \text{where} \ b_j \in A^{[k]} \quad \text{and} \quad \|a\|_{[k+1]} = \sum_{j=0}^\infty \|b_j\|_{[k]}.
\] (3.10)

By the definitions of \( \Phi_\xi \) and \( \Phi_{\xi'} \), we get

\[
p = \Phi_{\xi'}(a) = \sum_{n=0}^\infty \Phi_\xi(b_n) \xi_{k+1}^n \quad \text{in} \ \mathcal{H}_{\gamma(\xi')}.
\] (3.11)

By (3.10), the series \( \sum b_n \) converges absolutely in the Banach space \( A^{[k]} \). Since \( \Phi_\xi : A^{[k]} \to \mathcal{H}_\gamma \) is a continuous linear operator, the series \( \sum \Phi_\xi(b_n) \) converges absolutely in the Fréchet space \( \mathcal{H}_{\gamma(\xi)} \) and therefore in the Fréchet space \( \mathcal{H}_U \). Since \( \xi_{k+1}(U) \subseteq \mathbb{D} \), the series in (3.11) converges in \( \mathcal{H}_U \). Since \( U \) is open, connected and contains 0, the sum of the series in (3.11) and \( p \) coincide as functions on \( U \) by the uniqueness theorem: they are both holomorphic on \( U \) and have the same Taylor series at 0. Since \( q|\xi_{k+1} \), (3.11) implies that \( q|(p - \Phi_\xi(b_0)) \) in \( \mathcal{H}_U \). Since all zeros of \( q \) are in \( U \), \( q|(p - \Phi_\xi(b_0)) \) in \( \mathcal{H}_{\gamma(\xi)} \). By (3.9) and (3.10), \( \pi_{\xi,q}(p) \leq \|b_0\|_{[k]} \leq \|a\|_{[k+1]} \) since \( a \) is an arbitrary element of \( A^{[k+1]} \) satisfying \( \Phi_{\xi'}(a) = p \), (3.8) implies that \( \pi_{\xi,q}(p) \leq \pi_{\xi'}(p) \).

**Lemma 3.5.** Let \( q \in P_0, n \in \mathbb{N} \) and \( k > 0 \) be such that \( \deg q < n \). For every \( c > 0 \), let \( \delta(c) = (2kc)^{-1/n} \) and \( q_c = k(cz^n - q) \in P_0 \). Then for every sufficiently large \( c > 0 \), \( q_c(\delta(c)\mathbb{D}) \subseteq \mathbb{D} \) and all zeros of \( q_c \) belong to \( \delta(c)\mathbb{D} \).

**Proof.** Obviously, \( \lim_{c \to \infty} \delta(c) = 0 \). Since \( q(0) = 0 \), there is \( \alpha > 0 \) such that \( |q(z)| \leq \alpha |z| \) for all \( z \in \mathbb{D} \). Clearly, it suffices to show that \( q_c(\delta(c)\mathbb{D}) \subseteq \mathbb{D} \) and all zeros of \( q_c \) belong to \( \delta(c)\mathbb{D} \) whenever \( \delta(c) < \min\{1, \frac{1}{2\alpha}\} \).

Let \( c > 0 \) be such that \( \delta(c) < \min\{1, \frac{1}{2\alpha}\} \). If \( z \in \delta(c)\mathbb{D} \), then \( |kcz^n| = k|cz^n| = \frac{kc}{2\alpha} = \frac{1}{2} \) and \( |kq(z)| \leq k\alpha \delta(c) < \frac{k\alpha}{2\alpha} = \frac{1}{2} \). Hence \( |q_c(z)| \leq |kcz^n| + |kq(z)| < \frac{1}{2} + \frac{1}{2} = 1 \). Thus \( q_c(\delta(c)\mathbb{D}) \subseteq \mathbb{D} \).

Now if \( |z| = \delta(c) \), then \( |kcz^n| = k\alpha \delta(c) < \frac{k\alpha}{2\alpha} = \frac{1}{2} \), but \( |kq(z)| \leq k\alpha \delta(c) < \frac{k\alpha}{2\alpha} = \frac{1}{2} \). By the Rouché theorem [6], \( q_c = kcz^n - kq \) has the same number of zeros (counting with multiplicity) in \( \delta(c)\mathbb{D} \) as \( kcz^n \). The latter has \( n = \deg q_c \) zeros in \( \delta(c)\mathbb{D} \). Hence all the zeros of \( q_c \) are in \( \delta(c)\mathbb{D} \).

The proof of the next lemma is postponed until Section 4.
Lemma 3.6. Let \( k, \delta > 0, p \in \mathbb{P} \setminus \{0\} \) and \( m \in \mathbb{N} \). Then for every sufficiently large \( n \in \mathbb{N} \), there exists a connected open set \( W_n \subset \mathbb{C} \) such that \( 0 \in W_n \subseteq \delta \mathbb{D} \) and the polynomial \( q_n = kz((1+z)^n - p) \) has at least \( m \) zeros (counting with multiplicity) in \( W_n \) and satisfies \( q_n(W_n) \subseteq \mathbb{D} \).

Corollary 3.7. Let \( k > 0, p \in \mathbb{P} \setminus \{0\} \) and \( m \in \mathbb{N} \). Then there is \( n_0 \in \mathbb{N} \) and sequences \( \{W_n\}_{n \geq n_0} \) of connected non-empty open subsets of \( \mathbb{C} \) containing \( 0 \) and \( \{r_n\}_{n \geq n_0} \) of degree \( m \) polynomials such that for every \( r_n \to z^m \), \( \lim_{n \to \infty} \sup_{z \in W_n} |z| = 0 \), each \( r_n \) is a divisor of \( q_n = kz((1+z)^n - p) \), \( q_n(W_n) \subseteq \mathbb{D} \), and all zeros of \( r_n \) are in \( W_n \) for each \( n \geq n_0 \).

Proof. Applying Lemma 3.6 with \( \delta = 2^{-k} \) for \( k \in \mathbb{Z}_+ \), we find a strictly increasing sequence \( \{n_k\}_{k \in \mathbb{Z}_+} \) of positive integers such that for every \( k \in \mathbb{Z}_+ \) and every \( n \geq n_k \), there is a connected open subset \( W_{k,n} \subset \mathbb{C} \) for which

\[
0 \in W_{k,n} \subseteq 2^{-k} \mathbb{D}, \quad q_n(W_{k,n}) \subseteq \mathbb{D} \text{ and } q_n \text{ has at least } m \text{ zeros in } W_{k,n} \text{ for every } k \in \mathbb{Z}_+ \text{ and } n \geq n_k. \tag{3.12}
\]

The latter means that we can pick \( \lambda_{k,n,1}, \ldots, \lambda_{k,n,m} \in W_{k,n} \) such that \( r_{k,n} = \prod_{j=1}^{m} (z - \lambda_{k,n,j}) \) is a divisor of \( q_n \). Now for every \( n \geq n_0 \), we define \( r_n = r_{k,n} \) and \( W_n = W_{k,n} \) whenever \( n_k \leq n < n_{k+1} \). According to (3.12), each \( r_n \) is a divisor of \( q_n \), each \( r_n \) has all its zeros in \( W_n \), \( q_n(W_n) \subseteq \mathbb{D} \) and \( W_n \subseteq 2^{-k} \mathbb{D} \) provided \( n_k \leq n < n_{k+1} \). Hence \( \lim_{n \to \infty} \sup_{z \in W_n} |z| = 0 \) and \( r_n \to z^m \). \( \square \)

3.2 Proof of Theorem 1.2 modulo Lemma 3.6

Now we take Lemma 3.6 as granted and prove Theorem 1.2. Fix a sequence \( \{p_n\}_{n \in \mathbb{N}} \) in \( \mathbb{P}_0 \setminus \{0\} \) satisfying (3.1). We describe an inductive procedure of constructing sequences \( \{\xi_{k}\}_{k \in \mathbb{N}} \) in \( \mathbb{P}_0 \), \( \{c_{2k}\}_{k \in \mathbb{N}} \) of natural numbers and \( \{c_{2k}\}_{k \in \mathbb{N}} \) of positive numbers such that

(A0) \( \xi_1 = z \) and \( n_1 = 1 \);
(A1) \( \pi_{\xi_{[k]}}(z) > \frac{1}{2} \) for each \( k \in \mathbb{N} \), where \( \xi_{[k]} = (\xi_1, \ldots, \xi_k) \in \mathbb{P}_0^k \);
(A2) \( n_k > n_{k-1} \) for \( k \geq 2 \);
(A3) \( \xi_k = k(c_kz^{n_k} - p_{k/2}) \) for even \( k \geq 2 \) and \( \xi_k = k(z(1+z)^{n_k} - p(k-1)/2) \) for odd \( k \geq 3 \).

First, we take \( n_1 = 1, \xi_1 = z \) and observe that \( \pi_{\xi_{[1]}}(a_0 + a_1 z + \ldots + a_m z^m) = |a_0| + \ldots + |a_m| \). In particular, \( \pi_{\xi_{[1]}}(z) = 1 > \frac{1}{2} \). Thus (A0–A3) for \( k = 1 \) are satisfied and we have got the basis of induction. It remains to describe the induction step. Let \( k \geq 2 \) and \( \xi_j, n_j \) for \( j < k \) and \( c_j \) for \( j < k \) satisfying (A0–A3) are already constructed. We shall construct \( \xi_k, n_k \) and \( c_k \) (if \( k \) is even), satisfying (A1–A3).

Denote \( \gamma = \gamma(\xi_{[k-1]}) \). By Lemma 3.3, \( \pi_{\xi_{[k-1]}}(z^n) \to \pi(\xi_{[k-1]}) \) as \( n \to \infty \). By (A1) for \( k - 1 \), \( \pi_{\xi_{[k-1]}}(z) > \frac{1}{2} \). Hence we can pick \( m \in \mathbb{N} \) such that

\[
\pi_{\xi_{[k-1]}}(z^n) > \frac{1}{2} \text{ for every } n \geq m. \tag{3.13}
\]

Case 1: \( k \) is even. By (3.13), there is \( n_k \in \mathbb{N} \) such that \( n_k > \max\{n_{k-1}, \deg p_{k/2}\} \) and \( \pi_{\xi_{[k-1]}}(z^{n_k}) \) is \( \frac{1}{2} \). For \( c > 0 \), we consider the degree \( n_k \) polynomial \( q_c = k(cz^{n_k} - p_{k/2}) \in \mathbb{P}_0 \) and denote \( \delta(c) = (2kc)^{-1/n_k} \). Clearly, \( \delta(c) \to 0 \) as \( c \to \infty \). By Lemma 3.5,

\[
\delta(c) < \gamma, \quad q_c(\delta(c)\mathbb{D}) \subseteq \mathbb{D} \text{ and all zeros of } q_c \text{ are in } \delta(c)\mathbb{D} \text{ for all sufficiently large } c > 0. \tag{3.14}
\]

Since \( \frac{1}{\delta(c)c}q_c = z^{n_k} - \frac{1}{c}p_{k-1} \to z^{n_k} \text{ as } c \to \infty \), Lemma 3.3 implies that

\[
\pi_{\xi_{[k-1]}}(q_c(p)) = \pi_{\xi_{[k-1],\delta(c)}}(p) \to \pi_{\xi_{[k-1],z^{n_k}}}(p) \text{ as } c \to \infty \text{ for every } p \in \mathbb{P}. \tag{3.15}
\]

Using (3.15), (3.14) and the inequality \( \pi_{\xi_{[k-1]}}(z) > \frac{1}{2} \), we can choose \( c_k > 0 \) large enough in such a way that \( \delta = \delta(c_k) < \gamma \), all zeros of \( \xi_k = q_{c_k} = k(c_kz^{n_k} - p_{k/2}) \) are in \( \delta\mathbb{D}, \xi_k(\delta\mathbb{D}) \subseteq \mathbb{D} \) and \( \pi_{\xi_{[k-1]}}(\xi_k(z)) > \frac{1}{2} \). By
Lemma 3.4. \( \pi_{\xi_k}(p) \geq \pi_{\xi_{k-1}}(p) \) for every \( p \in \mathbb{P} \). In particular, \( \pi_{\xi_k}(z) \geq \pi_{\xi_{k-1}}(z) > \frac{1}{2} \). It remains to notice that (A1–A3) are satisfied.

Case 2: \( k \) is odd. By (3.13), \( \pi_{\xi_{k-1}}(z) \geq \frac{1}{2} \). By Corollary 3.7, there is \( l \in \mathbb{N} \) and sequences \( \{W_n\}_{n \geq l} \) of connected non-empty open subsets of \( \mathbb{C} \) containing 0 and \( \{r_n\}_{n \geq l} \) of degree \( m \) polynomials such that \( r_n \rightarrow z^m, \lim_{n \rightarrow \infty} \sup_{z \in W_n} |z| = 0 \), each \( r_n \) is a divisor of \( q_n = k(z(1+z)^n - p_{(k-1)/2}) \), \( q_n(W_n) \subseteq \mathbb{D} \) and all zeros of \( r_n \) are in \( W_n \) for each \( n \in \mathbb{N} \). By Lemma 3.3, \( \pi_{\xi_{k-1}}(z) \geq \frac{1}{2} \) as \( n \rightarrow \infty \) and therefore we can pick \( n_k > \max\{l, n_{k-1}\} \) such that \( \pi_{\xi_{k-1}}(z) > \frac{1}{2} \) and \( W_{n_k} \subseteq \gamma \mathbb{D} \). Put \( \xi_k = q_{n_k} = k(z(1+z)^{n_k} - p_{(k-1)/2}) \).

By Lemma 3.4, \( \pi_{\xi_k}(z) \geq \pi_{\xi_{k-1}}(z) > \frac{1}{2} \). It remains to notice that (A1–A3) are again satisfied.

This concludes the inductive construction of the sequences \( \{\xi_k\}_{k \in \mathbb{N}}, \{n_k\}_{k \in \mathbb{N}} \) and \( \{c_{2k}\}_{k \in \mathbb{N}} \) satisfying (A0–A3). By Lemma 3.4, \( \pi_{\xi_{k+1}}(p) \leq \pi_{\xi_{k}}(p) \) for every \( p \in \mathbb{P} \). Thus, \( \{\pi_{\xi_k}\}_{k \in \mathbb{N}} \) is a pointwise decreasing sequence of submultiplicative norms on \( \mathbb{P} \). Hence the formula \( \pi(p) = \lim_{k \rightarrow \infty} \pi_{\xi_k}(p) \) defines a submultiplicative seminorm on \( \mathbb{P} \). By (A1), \( \pi_{\xi_k}(z) > \frac{1}{2} \) for each \( k \in \mathbb{N} \) and therefore \( \pi(z) \geq \frac{1}{2} > 0 \). Hence \( \pi \) is non-zero. From (3.8) it immediately follows that \( \pi_{\xi_k}(\xi_k) \leq 1 \) for every \( k \in \mathbb{N} \). Indeed, \( \|u_k\|_{[k]} = 1 \) and \( \phi_{\xi_k}(u_k) = \xi_k \).

Hence \( \pi(\xi_k) \leq \pi_{\xi_k}(\xi_k) \leq 1 \). By (A3), \( 2\xi k = 2k(c_{2k}z^{n_{2k}} - \pi_k) \) for each \( k \in \mathbb{N} \). Hence \( \pi(c_{2k}z^{n_{2k}} - \pi_k) \leq \frac{1}{2\xi k} \) for every \( k \in \mathbb{N} \) and therefore \( \pi(c_{2k}z^{n_{2k}} - \pi_k) \) for \( k \in \mathbb{N} \). Hence \( \pi(z(1+z)^{n_{2k+1}} - \pi_k) \leq \frac{1}{2\xi k} \) for each \( k \in \mathbb{N} \) and therefore \( \pi(z(1+z)^{n_{2k+1}} - \pi_k) \) for each \( k \in \mathbb{N} \). Thus all conditions of Lemma 3.1 are satisfied. By Lemma 3.1, the restriction of \( \pi \) to \( \mathbb{P} \) is a submultiplicative norm on \( \mathbb{P} \) and the completion of the normed algebra \( (\mathbb{P}, \pi) \) is an infinite dimensional chaotic Banach algebra with \( z \) being a chaotic element. The proof of Theorem 1.2 modulo Lemma 3.6 is complete.

## 4 Proof of Lemma 3.6

Our main instrument is the argument principle [6]. We recall the related basic concepts. An oriented path \( \Gamma \) in \( \mathbb{C} \) with the source \( s(\Gamma) \) and the end \( e(\Gamma) \) is a set of the shape \( \Gamma = \varphi([a, b]) \), where \( \varphi : [a, b] \rightarrow \mathbb{C} \) is continuous, \( \varphi(a) = s(\Gamma) \), \( \varphi(b) = e(\Gamma) \) and \( \varphi_{|[a,b]} \) is injective. Such a map \( \varphi \) is a parametrization of the path \( \Gamma \). The oriented path \( \Gamma \) is closed if \( s(\Gamma) = e(\Gamma) \). If \( \Gamma \) is an oriented path in \( \mathbb{C} \) such that \( \Gamma \rightarrow \mathbb{C} \setminus \{0\} \) is continuous, we can find continuous \( \varphi : [a, b] \rightarrow \Gamma \) and \( \psi : [a, b] \rightarrow \mathbb{R} \) such that \( \varphi(a) = s(\Gamma), \varphi(b) = e(\Gamma) \) and \( \frac{d\varphi(t)}{dt} = e^{i\psi(t)} \) for every \( t \in [a, b] \). The number \( \frac{\psi(b) - \psi(a)}{2\pi i} \) does not depend on the choice of \( \varphi \) and \( \psi \) and is called the winding number of \( f \) along the path \( \Gamma \) and denoted \( w(f, \Gamma) \). Alternatively, \( 2\pi w(f, \Gamma) \) is the variation of the argument of \( f \) along \( \Gamma \).

We need few well-known properties of the winding numbers. If \( \Gamma \) and \( \Gamma' \) are two non-closed oriented paths with \( e(\Gamma) = s(\Gamma') \) and \( \Gamma \setminus \{e(\Gamma), s(\Gamma')\} \cap (\Gamma' \setminus \{e(\Gamma'), s(\Gamma')\}) = \emptyset \), then \( \Gamma \cup \Gamma' \) can be naturally considered as an oriented path with the source \( s(\Gamma) \) and the end \( e(\Gamma') \). Then

\[
w(f, \Gamma \cup \Gamma') = w(f, \Gamma) + w(f, \Gamma') \quad \text{for each continuous } f : \Gamma \cup \Gamma' \rightarrow \mathbb{C} \setminus \{0\}.
\]

(4.1)

Variants of the following elementary property exist in the literature under different names, one of which is the dog on a leash lemma. If \( \Gamma \) is an oriented path in \( \mathbb{C} \) and \( f, g : \Gamma \rightarrow \mathbb{C} \) are continuous, then

\[
|w(f + g, \Gamma) - w(f, \Gamma)| < 1/2 \quad \text{if } |g(z)| < |f(z)| \text{ for each } z \in \Gamma.
\]

(4.2)

It is easy to see that if \( \Gamma \) is an oriented path, \( f : \Gamma \rightarrow \mathbb{C} \setminus \{0\} \) is continuous and \( |w(f, \Gamma)| \geq n/2 \) with \( n \in \mathbb{N} \), then \( f \) crosses every line in \( \mathbb{C} \) passing through 0 at least \( n \) times. In other words, if \( c \in \mathbb{T} \), then

\[
|w(f, \Gamma)| < \frac{n+1}{2} \quad \text{if } \{z \in \Gamma : f(z) \in c\mathbb{R}\} \text{ consists of at most } n \text{ points}.
\]

(4.3)

We use the above property to prove the following lemma.

**Lemma 4.1.** If the oriented path \( \Gamma \) in \( \mathbb{C} \) is an interval of a straight line, \( f \) is a polynomial of degree at most \( m \in \mathbb{Z}_+ \) and \( g : \Gamma \rightarrow \mathbb{C} \) is a continuous map taking values in a line in \( \mathbb{C} \) passing through zero such that \( f(z) + g(z) \neq 0 \) for every \( z \in \Gamma \), then \( w(f + g, \Gamma) < \frac{m+1}{2} \).
Proof. Since $\Gamma$ is an interval of a straight line we can parametrize $\Gamma$ by $\varphi : [0,1] \to \mathbb{C}$, $\varphi(t) = at + b$ with $a,b \in \mathbb{C}$, $a \neq 0$. Since $g$ takes values in a line in $\mathbb{C}$ passing through zero, there is $c \in \mathbb{T}$ such that $g(z) \in c^{-1}\mathbb{R}$ for $z \in \Gamma$. Since the function $h(t) = \text{Im} cf(at + b)$ is a polynomial with real coefficients of degree at most $m$, it either vanishes identically on $[0,1]$ or has at most $m$ zeros on $[0,1]$.

If $h \equiv 0$, then $f + g : I \to \mathbb{C}$ takes values in the line $c^{-1}\mathbb{R}$. Hence $w(f + g, \Gamma) = 0 < \frac{m+1}{2}$. If $h \neq 0$, then the set $C = \{ t \in [0,1] : h(t) = 0 \}$ consists of at most $m$ points. It is easy to see that the set $C' = \{ z \in \Gamma : (f + g)(z) \in c^{-1}\mathbb{R} \}$ coincides with $\{at + b : t \in C\}$ and therefore $C'$ consists of at most $m$ points. By (4.3), $w(f + g, \Gamma) < \frac{m+1}{2}$.

Finally, we remind the argument principle.

**Argument Principle.** Let $U$ be a bounded open subset of $\mathbb{C}$, whose boundary is a closed oriented path $\Gamma$, which encircles $U$ counterclockwise. Let also $f : \mathbb{U} \to \mathbb{C}$ be a continuous function such that $f$ is holomorphic on $U$ and $0 \notin f(\Gamma)$. Then $w(f, \Gamma)$ is exactly the number of zeros of $f$ in $U$ counted with multiplicity.

We are ready to prove Lemma 3.6. Let $k, \delta > 0$, $p \in \mathbb{P} \setminus \{0\}$ and $m \in \mathbb{N}$. We have to show that for every sufficiently large $n \in \mathbb{N}$, there exists a connected open set $W_n \subset \mathbb{C}$ such that $0 \in W_n \subset \delta \mathbb{D}$ and the polynomial $q_n = k z ((1 + z)^n - p)$ has at least $m$ zeros in $W_n$ and satisfies $q_n(W_n) \subset \mathbb{D}$.

Since at most one of the polynomials $q_n$ can be zero, there is $n_0 \in \mathbb{N}$ such that $q_n \neq 0$ for $n \geq n_0$. Let $c > 1$ be such that $|p(z)| \leq c$ for every $z \in \mathbb{D}$. Pick $\alpha \in (0,1)$ such that $\alpha < \delta$, $\alpha < \frac{1}{\sqrt{c}}$, the circle $(\sin \alpha) \mathbb{T}$ contains no zeros of $p$ and the rays $\{-1 + te^{i\alpha} : t > 0\}$ and $\{-1 + te^{-i\alpha} : t > 0\}$ contain no zeros of $q_n$ for every $n \geq n_0$. For every $n \in \mathbb{N}$, let $\varepsilon_n = (2e)^{1/n}$. Clearly $\{\varepsilon_n\}$ is a strictly decreasing sequence of positive numbers convergent to 1. Now for each $n \in \mathbb{N}$, we consider the open set $W_n \subset \mathbb{C}$ defined by the formula:

$$W_n = \{-1 + r e^{i\beta} : -\alpha < \beta < \alpha, \cos \beta - \sqrt{\cos^2 \beta - \cos^2 \alpha} < r < \varepsilon_n\}.$$

It is easy to see that $W_n$ is convex and therefore connected, open and contains 0. The following picture shows the set $W_n$.

![Diagram showing the set $W_n$](image)

The boundary $\partial W_n$, oriented in such a way that it encircles $W_n$ counterclockwise, is the concatenation of 4 oriented paths $\partial W_n = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ defined above. Clearly $\Gamma_1$ is an arc of the circle $-1 + \varepsilon_n \mathbb{T}$, $\Gamma_3$ is an arc of the circle $(\sin \alpha) \mathbb{T}$, while $\Gamma_2$ and $\Gamma_4$ are intervals of the straight lines $-1 + e^{i\alpha} \mathbb{R}$ and $-1 + e^{-i\alpha} \mathbb{R}$ respectively. In each case the parametrization is chosen to agree with the right orientation. First, observe that the farthest from 0 points of $\partial W_n$ are $B = -1 + \varepsilon_n e^{i\alpha}$ and $A = -1 + \varepsilon_n e^{-i\alpha}$. Hence $W_n$ is contained in the disk $| -1 + \varepsilon_n e^{i\alpha} | \mathbb{D}$. Since $| -1 + \varepsilon_n e^{i\alpha} | \to | -1 + e^{i\alpha} | = 2 \sin \frac{\alpha}{2} < \alpha$ as $n \to \infty$, we have

$$W_n \subset \alpha \mathbb{D} \subset \delta \mathbb{D}$$

for each sufficiently large $n$. (4.4)

Since $\alpha < 1$, we also have $W_n \subset \mathbb{D}$ for $n$ large enough. Since $|p(z)| \leq c$ for $z \in \mathbb{D}$, $|(1 + z)^n| \leq 2c$ for $z \in -1 + \varepsilon_n \mathbb{D}$ and $W_n \subset -1 + \varepsilon_n \mathbb{D}$, we see that $(1 + z)^n - p(z) \leq 3c$ for all $z \in W_n$ for all sufficiently large
Since $\alpha < \frac{1}{6k}$ and $\sup_{z \in W_n} |z| < \alpha$ for all $n$ large enough, we have $|q_n(z)| < k\alpha|(1 + z)^n - p(z)| \leq 3ck\alpha < 1$ for $z \in W_n$ for all sufficiently large $n$. Hence

$$q_n(W_n) \subseteq \mathbb{D} \text{ for each sufficiently large } n.$$ (4.5)

According to (4.4) and (4.5), it suffices to show that $r_n = (1 + z)^n - p$ has at least $m$ zeros in $W_n$ for each sufficiently large $n$. Since $r_n$ have no zeros on the rays $\{-1 + te^{i\alpha} : t > 0\}$ and $\{-1 + te^{-i\alpha} : t > 0\}$ for every $n \geq n_0$, $r_n$ have no zeros on $\Gamma_1 \cup \Gamma_4$ for all $n$ large enough. Since $|(1 + z)^n| = 2c$ for $z \in \Gamma_1$ and $|p(z)| \leq c$ for $z \in \Gamma_1 (\Gamma_1 \subset \mathbb{D}$ for $n$ large enough), we see that $r_n(z) \neq 0$ for $z \in \Gamma_1$ for all sufficiently large $n$. Since $\Gamma_2 \subset (\sin \alpha)\mathbb{T}$ and $p$ has no zeros on the circle $(\sin \alpha)\mathbb{T}$, we have $|p(z)| = c_0 > 0$. It is easy to see that $\Gamma_3$ does not depend on $n$ and is a compact subset of the disk $-1 + \mathbb{D}$. Hence $(1 + z)^n$ converges uniformly to 0 on $\Gamma_3$ as $n \to \infty$. Thus $|p(z)| > |(1 + z)^n|$ and therefore $r_n(z) \neq 0$ for $z \in \Gamma_3$ for all $n$ large enough.

By the argument principle and (4.1), the number $\nu(n)$ of zeros of $r_n$ in $W_n$ satisfies

$$\nu(n) = w(r_n, \partial W_n) = \sum_{j=1}^{4} w(r_n, \Gamma_j) \text{ for all sufficiently large } n.$$ (4.6)

Since on each of $\Gamma_2$ and $\Gamma_4$, the function $(1 + z)^n$ takes values in a line in $\mathbb{C}$ passing through zero and $\Gamma_2$ and $\Gamma_4$ are intervals of straight lines, Lemma 4.1 implies that

$$|w(r_n, \Gamma_2)| < \frac{\deg p + 1}{4} \text{ and } |w(r_n, \Gamma_4)| < \frac{\deg p + 1}{4} \text{ for every sufficiently large } n.$$ (4.7)

Since $|(1 + z)^n| < |p(z)|$ for $z \in \Gamma_3$ for any $n$ large enough, (4.2) implies that

$$|w(r_n, \Gamma_3)| < |w(p, \Gamma_3)| + \frac{1}{2} \text{ for every sufficiently large } n.$$ (4.8)

Finally, since $|p(z)| < |(1 + z)^n|$ for $z \in \Gamma_1$ for any $n$ large enough, (4.2) implies that

$$w(r_n, \Gamma_1) > w((1 + z)^n, \Gamma_1) - \frac{1}{2} \text{ for every sufficiently large } n.$$ (4.9)

A direct computation shows that $w((1 + z)^n, \Gamma_1) = 2n\alpha$. Hence by the last display,

$$w(r_n, \Gamma_1) > 2n\alpha - \frac{1}{2} \text{ for every sufficiently large } n.$$ (4.9)

Combining (4.6–4.9), we get

$$\nu(n) > 2n\alpha - 2 - |w(p, \Gamma_3)| - \deg p \text{ for every sufficiently large } n.$$ (4.9)

Since $\Gamma_3$ does not depend on $n$, $\nu(n) \to \infty$ as $n \to \infty$. Hence $r_n$ and therefore $q_n$ has at least $m$ zeros in $W_n$ for each $n$ large enough. The proof of Lemma 3.6 and that of Theorem 1.2 is complete.

5 Remarks and open questions

1. Our construction of a chaotic Banach algebra provides little control over its Banach space structure. Thus the following interesting questions arise.

**Question 5.1.** Which separable infinite dimensional Banach spaces admit a multiplication turning them into a supercyclic or into an almost hypercyclic Banach algebra? In particular, is there a multiplication on $\ell_2$, turning it into a chaotic Banach algebra?
2. The structural properties of the class of supercyclic or almost hypercyclic Banach algebras remain a complete mystery.

3. Let $\mathcal{H}$ be the Hilbert space of Hilbert–Schmidt operators on $\ell_2$. With respect to the composition multiplication, $\mathcal{H}$ is a non-commutative non-unital Banach algebra. Let also $S \in \mathcal{H}$ be defined by its action on the basic vectors as follows: $Se_0 = 0$, $Se_n = n^{-1}e_{n-1}$ if $n \geq 1$. Consider the left multiplication by $S$ operator $\Phi \in L(\mathcal{H})$, $\Phi(T) = ST$. Using the hypercyclicity and supercyclicity criteria [1], it is easy to see that $\Phi$ is supercyclic and $I + \Phi$ is hypercyclic. Thus supercyclicity of a multiplication operator and hypercyclicity of a perturbation of the identity by a multiplication operator on a non-commutative Banach algebra is a much simpler phenomenon.

4. We would also like to raise the following question. We say that a Banach algebra $A$ is wildly chaotic if it has a supercyclic element $a$ such that for every $z \in \mathbb{T}$, the set $\{a(z + a)^n : n \in \mathbb{N}\}$ is dense in $A$.

**Question 5.2.** Does there exist a wildly chaotic infinite dimensional Banach algebra?

Note that our construction can be modified to make $\{a(z + a)^n : n \in \mathbb{N}\}$ dense in $A$ for each $z$ from a given countable subset of $\mathbb{T}$.

5. Corollary 1.5 ensures the existence of a hypercyclic operator $T$ with $\sigma(T) = \{1\}$ and $T \oplus T$ being non-cyclic. This naturally leads to the question whether such operators exist on every separable infinite dimensional Banach space.

**Question 5.3.** Let $X$ be a separable infinite dimensional Banach space. Does there exist a $T \in L(X)$ such that $T$ is hypercyclic, $T \oplus T$ is non-cyclic and $\sigma(T) = \{1\}$? What is the answer for $X = \ell_2$?

The above question is related to the following question of Bayart and Matheron [2].

**Question 5.4.** Does every separable infinite dimensional Banach space admit a hypercyclic operator $T$ such that $T \oplus T$ is non-cyclic?

6. Bayart and Matheron [1] ask whether there exists a hypercyclic strongly continuous operator semigroup $\{T_t\}_{t \geq 0}$ on a Banach space $X$ such that the semigroup $\{T_t \oplus T_t\}_{t \geq 0}$ acting on $X \oplus X$ is non-hypercyclic. As we have already mentioned, Theorem 1.2 provides a quasinilpotent operator $M_a$ on the Banach space $A$ such that $I + M_a$ is hypercyclic, while $(I + M_a) \oplus (I + M_a)$ is non-hypercyclic. Since $M_a$ is quasinilpotent,

$$S = \ln(I + M_a) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} M_a^n$$

is a well-defined (also quasinilpotent) continuous linear operator on $A$. Hence we can consider the operator norm continuous semigroup $\{e^{tS}\}_{t \geq 0}$, which contains all powers of $I + M_a$: $e^{ntS} = (I + M_a)^n$ for $n \in \mathbb{N}$. It follows that $\{e^{tS}\}_{t \geq 0}$ is hypercyclic. On the other hand, $e^{tS} \oplus e^{tS} = (I + M_a) \oplus (I + M_a)$ is a non-hypercyclic member of the semigroup $\{e^{tS} \oplus e^{tS}\}_{t \geq 0}$. According to Conejero, Müller and Peris [3], $T_t$ is hypercyclic for every $t > 0$ if $\{T_t\}_{t \geq 0}$ is a hypercyclic strongly continuous operator semigroup. Hence $\{e^{tS} \oplus e^{tS}\}_{t \geq 0}$ is non-hypercyclic which answers negatively the above mentioned question of Bayart and Matheron.

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**References**


