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Dynamical symmetry breaking with optimal control: Reducing the number of pieces

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I. INTRODUCTION

Nonequilibrium dynamics of many-body systems have been the subject of intensive investigation in statistical physics. While for systems in quasistatic equilibrium, fluctuation-dissipation relations can be applied, these in general do not hold for systems driven out of equilibrium.1 Interest in the dissipation relations can be applied, these in general do not...

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the symmetric vacuum in the disordered phase. When \( \epsilon(t) > 0 \) the potential \( V(\varphi) \) is characterized by two symmetry-broken minima of the ordered phase \( \varphi = \pm \sqrt{\epsilon} \). The critical point thus corresponds to \( \epsilon(t) = 0 \). The order parameter dynamics in space and time fulfills the Ginzburg-Landau partial differential equation:

\[
\left[ \frac{\partial^2}{\partial t^2} + \eta \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right] \varphi(x,t) + \frac{\partial V(\varphi)}{\partial \varphi} = \vartheta(x,t), \tag{2}
\]

where \( \eta \) and \( \vartheta(x,t) \) are the phenomenological dissipation rate and Langevin force, respectively, that ensure thermalization for constant \( \epsilon(t) \). In this paper we will consider dimensionless units such that \( \eta = 1 \). Model (2) has been employed by Laguna and Zurek to verify numerically Kibble-Zurek scaling, in the simplest possible scenario.\(^{23} \)

When \( \epsilon(t) \) is changed rapidly in time from a negative to a positive value, the order parameter exhibits spontaneous local decay towards either the positive or negative minimum of \( V(\varphi) \). Crucially, in spatially separated regions, the order parameter \( \varphi(x,t) \) may develop an opposite sign giving rise to defects.

We assume the Langevin forces to be random variables with no spatial or temporal correlations:

\[
\langle \vartheta(x,t)\vartheta(x',t') \rangle = 2\eta \theta(x-x') \delta(t-t'), \tag{3}
\]

where \( \theta \) is an effective temperature of the environment that is in contact at all times with the system. In accordance with Laguna and Zurek\(^{23} \) we choose \( \theta = 0.01 \). This low temperature value ensures that the density of defects in the form of domain walls that might arise from thermal fluctuations for \( \epsilon > 0 \) is negligible.\(^{31} \) This means that practically all the defects that we count at the end of the process are formed during the fast quench of \( \epsilon(t) \). We also assume the system to be in the overdamped regime, corresponding to the parameter \( \eta \) being larger than all the real eigenfrequencies of Eq. (2). Under this assumption, the order parameter will always monotonically decay to its steady state when \( \epsilon \) ceases to change.

For our numerical simulations, we employ the finite-difference method and the velocity Verlet algorithm to simulate the dynamics of Eq. (2). As in Ref. 23, we initially take \( N = 2^{14} \) grid points with a periodic domain. This relatively large number of points allows us to recover in a clear and unambiguous way the Kibble-Zurek scaling. The initial condition is \( \varphi(x,t_{in}) = 0 \), where \( t_{in} \) is the initial time.

During the quench protocol, \( \epsilon(t) \) changes from \( \epsilon(t_{in}) = -2 \) at the initial time \( t_{in} \) to \( \epsilon(t_{fin}) = 5 \) at the final time \( t_{fin} \) such that the total time is \( T = t_{fin} - t_{in} \). For this choice of the initial and final values of \( \epsilon \), the average value of the equilibrium order parameter coincides with the minimum of the potential energy (1) with only small fluctuations. For each simulation, we count the number of defects \( N_D \) as the number of zeros of the order parameter \( \varphi(x,t_{fin}) \) (counting the pairs of adjacent grid points where \( \varphi \) changes sign). We average \( N_D \) over no less than \( N_{rev} = 10^3 \) different realizations of the Langevin forces. This is enough to obtain small statistical fluctuations in the average results.

### III. Linear and Nonlinear Quenches

We begin our investigation by testing our model and its finite-difference implementation for linear and nonlinear quenches of the form

\[
\epsilon(t) = \frac{t}{|t|} \left( \frac{|t|}{\tau_Q} \right) ^{\alpha}, \tag{4}
\]

where we defined the rate \( \tau_Q^{-1} \) of crossing the critical point. Notice that the critical point \( \epsilon = 0 \) is always reached at \( t = 0 \). In order to ensure the correct initial and final values we set

\[
t_{in} = -2^{1/\alpha} \tau_Q, \quad t_{fin} = 5^{1/\alpha} \tau_Q. \tag{5}
\]

Therefore, with these settings, the total time \( T = t_{fin} - t_{in} \) depends on both \( \alpha \) and \( \tau_Q \). A few examples of the time dependence of \( \epsilon(t) \) are shown in Fig. 1.

For the protocol in Eq. (4), the predicted scaling for the average number of defects reads\(^{24—26} \)

\[
N_D \sim \left( \frac{1}{\tau_Q} \right)^{\chi}, \tag{6}
\]

\[
\chi = \frac{a v}{\alpha \mu + 1}, \tag{7}
\]

where, for the Ginzburg-Landau model we consider, \( v = 1/2 \) and \( \mu = 1 \) are the correlation length and relaxation time critical exponents of the mean-field universality class. For \( \alpha = 1 \), Eq. (7) gives \( \chi = 1/4 \) as first derived by Zurek.\(^{5} \)

After performing numerical simulations of Eq. (2) with \( \alpha = 1 \) and measuring the average number of defects \( N_D \), we find the results shown in Fig. 2 as a function of the quench time \( \tau_Q \). The scaling of \( N_D \) with \( \tau_Q^{-1} \) is linear over more than 2 order of magnitudes thanks to the large size of the integration domain. The best-fit result, in the linear region, for the scaling exponent is \( \chi_{fit} = 0.258 \pm 0.004 \) that is very close to the expected result \( \chi = 0.25 \).

![Figure 1](attachment:image.png)
section we will compare predictions (8) and (9) with our numerical simulations, and in the following we will show that by employing optimal control techniques we can reduce the number of defects even further.

In order to compare our numerical simulations with the predictions of Ref. 28, we modified the quench function $\varepsilon(t)$ of Eq. (4) so that the total quenching time $T$ for going from $\varepsilon(t_{\text{in}}) = -2$ to $\varepsilon(t_{\text{in}}) = 5$ is fixed a priori. In this setting, the function Eq. (4) is still valid, but the corresponding quench rate $\tau_Q$ now depends on $T$ and $\alpha$:

$$\tau_Q = \frac{T}{2^{1/\alpha} + 5^{1/\alpha}}. \tag{10}$$

The expressions for $t_{\text{in}}$ and $t_{\text{fin}}$ of Eq. (5) remain unchanged.

For $T = 20$, 40, 60, 80 we vary $\alpha$ and compute the average number of defects $N_D$. The results are shown in Fig. 3. For small values of $T$ we observe a clear optimal value $\alpha$ where the number of defects $N_D$ are minimized. As $T$ increases the minimum is very shallow and for $T > 80$ we do not observe any clear minimum and $N_D$ decays to an asymptotic value. To find the optimal values $\alpha_{\text{opt}}$ and $N_{D,\text{opt}}$ we interpolate the data with cubic splines. The estimates thus obtained are illustrated in Fig. 4.

In the top panel we show the estimated $\alpha_{\text{opt}}$ as a function of $T$ in a semilogarithmic scale. The data points show a clear linear behavior, thus we fit them with a simplified fitting function:

$$\alpha_{\text{opt}} = A \ln(C T). \tag{11}$$

We are therefore assuming that for a limited range of time lapses $T$, the double logarithmic term in Eq. (8) can be neglected. After fitting the data, we extract the estimate for the prefactor $A \simeq 1.8 \pm 0.1$. This is quite in disagreement with the expected result $1/\mu = 1$. The full model of Eq. (8) would not give a straight line in this scale and in fact does not agree with our numerical simulations. In the small range of values of $T$ we were able to analyze, $\alpha_{\text{opt}}$ is well described by a power law of the total time $T$. We believe that the full model of Eq. (8) would be more appropriate for larger values of $T$. However, in our numerical simulations, as we show in Fig. 3, we cannot take larger values of $T$ as it is very hard for us to accurately identify a minimum.

B. Optimal number of defects $N_{D,\text{opt}}$

We now turn to the analysis of the optimized average number of defects. The results of the numerical calculations are shown in the bottom panel of Fig. 4. As we would like to compare the numerical data with the prediction of Eq. (9), we plot the data in log-log scale. As before, we observe that the data show a clear linear scaling and therefore we fit them with a simple power law:

$$\tilde{N}_{D,\text{opt}} = \left[\frac{1}{C T}\right]^{\xi}. \tag{12}$$

The fitting gives the estimate $\xi \simeq 0.503 \pm 0.005$, which is in strong agreement with the prediction $\nu/\mu = 0.5$ from Ref. 28. Therefore our numerical data for the average number of defects is well described by theoretical scaling relations. In contrast to the data for $\alpha_{\text{opt}}$, we find that $N_{D,\text{opt}}$ is less sensitive to the limited range of $T$. 

![FIG. 2. (Color online) Scaling of the average number of defects $N_D$ (points) as a function of $1/\tau_Q$ in log-log scale for $\alpha = 1$ (top) and $\alpha = 4$ (bottom). Also shown are the best-fit lines according to prediction (6).](image-url)
IV. OPTIMIZED QUENCHES

In this section we want to find strategies to minimize the production of defects, for a fixed time $T$ for crossing the phase transition, by tailoring the time dependence of the reduced temperature $\epsilon(t)$. We go beyond the simple power-law dependence presented in Eq. (4) and add a correction term $f(t)$ to it:\(^{32}\)

$$
\epsilon(t) = \frac{t}{|t|} \left( \frac{|t|}{\tau_Q} \right)^{\alpha} [1 + f(t)].
$$

(13)

We require that $|f(t_{in})|, |f(t_{fin})| \ll 1$ so that the initial and final values of the reduced temperature $\epsilon$ coincide approximately with the previously used values. Our task is then to find the function $f(t)$ that reduces the average number of defects created. This is a typical problem of optimal control (see, for example, Ref. 33) that has been recently employed for efficient cooling of many-body systems.\(^{34}\) There are many algorithms that can be employed for this task and that could in principle guarantee monotonic decrease of the target cost function, in this case the average number of defects. We, however, use a simple yet powerful procedure inspired by the CRAB algorithm that was designed originally for optimizing the dynamics of many-body quantum systems.\(^{30}\) The basic idea is to decompose the correction $f(t)$ as a linear superposition of trigonometric functions:

$$
\begin{equation}
\begin{aligned}
f(t) &= \frac{1}{\lambda(t)} \sum_{n=1}^{n_{max}} A_n \cos \omega_n t + B_n \sin \omega_n t,
\end{aligned}
\end{equation}
$$

(14)

where $n_{max}$ is the total number of frequencies $\omega_n$ that generate the correction $f(t)$; $A_n$ and $B_n$ are the amplitudes of the oscillating terms and we impose the following constraints:

$$
A_n^2 \leq 1, \ B_n^2 \leq 1,
$$

(15)

which ensures that the optimization algorithm will not yield oscillating functions with large amplitudes; finally the function $\lambda(t)$ forces the correction function to be smooth at the boundaries $t_{in}$ and $t_{fin}$. Although the specific form of $\lambda(t)$ is not crucial for the optimization, we use the function

$$
\lambda(t) = 1 + \Lambda \left[ e^{-|t-t_{in}|^2} + e^{-|t-t_{fin}|^2} \right]
$$

(16)

with the parameter $\Lambda = 100$ forcing the control function $f(t)$ to be very small at the two endpoints. For the frequencies appearing in Eq. (15), we choose

$$
\omega_n = \frac{2\pi n}{T}.
$$

(17)

We first considered $T = 20$ for concreteness. From the analysis in Sec. III A, we know that the best exponent for the nonlinear quench for $T = 20$ is $\alpha = 0.6$. This setting gives an
average number of defects of $N_D \simeq 128 \pm 1$. We used standard Matlab minimization routines to find the best values $A_n$ and $B_n$. The results are summarized in Fig. 6.

The best result is for $n_{\text{max}} = 5$, corresponding to 10 optimization parameters, yielding $N_D = 81.9 \pm 0.2$ that is more than 40% less than the nonoptimized result. For larger $n_{\text{max}}$ we cannot find better results as the number of free parameters is too large for the optimization routines.

The resulting optimized time dependence of the reduced temperature $\varepsilon(t)$ is shown in Fig. 5 and compared with the function $\varepsilon(t)$ without optimization. Similar to other optimal control results, the control function exhibits nonadiabatic oscillations that eventually lead to a reduction of the number of defects. While in the quantum scenario, as for example analyzed in Ref. 35, this is ascribed to constructive interference of many paths leading to the desired target state, in our classical model this might be interpreted as constructive interference of classical waves reducing the number of defects created. It is interesting to notice that the reduced temperature does not change monotonically and actually oscillates around zero a number of times. In terms of the Landau potential Eq. (1), the system evolves back and forth from a potential with a single minimum at zero order parameter $\phi = 0, \varepsilon < 0$ to a potential with two minima $\phi \neq 0, \varepsilon > 0$.

We have extended our analysis to different system sizes $N$ ranging from $2^8$ to $2^{14}$ and also to a different total time $T = 10$. The latter results have been obtained optimizing the nonlinear quench (13) with $\alpha = 0.5$. The results for the density of defects $n_D = N_D/N$ are shown in Fig. 6. The data reveal that the optimized pulses are not very sensitive to the size of the system. Therefore, the performance of our optimization
Our optimization protocol does not depend strongly on the exact number of particles in the system. Our optimization protocol is also quite robust to small imperfections in the coefficients $A$ and $B$. After perturbing these coefficients by random time-independent fluctuations of magnitude smaller than 1% we find, on average, an increase in the number of defects by 3%.

V. CONCLUSIONS

In summary, we have provided numerical evidence that the total number of defects created during the crossing of a second-order phase transition can be effectively reduced by appropriately tailoring the time dependence of the reduced temperature $\tilde{\varepsilon}(t)$. This optimization is only valid for a finite system, which is the relevant case for experiments. In the thermodynamical limit, the results presented in Ref. 28 should remain valid: The optimal time dependence in the vicinity of the critical point should be a power law with an exponent $\alpha$ fulfilling universal scaling relations.

Our optimization takes place in an open-system scenario, in which the system is always in contact with a thermal reservoir. In our simulations, this is embodied by the Langevin forces and the friction term. It is thus remarkable that a simple and intuitive technique as CRAB works in this nonideal case. Moreover, as the number of frequencies is kept small, the bandwidth of the control function $f(t)$ can be kept under control for a realistic implementation.

Finally, we would like to stress that our work could be applied in experiments with classical systems undergoing 1D structural phase transitions of the second order such as those occurring for cold ions in highly anisotropic traps.15

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32The form in Eq. (13) implies that $\varepsilon(0) = 0$ regardless of the optimizing function. We have also tried the following ansatz that relaxes the previous constraint: $\varepsilon(t) = \frac{1}{\tau_{Q}} \left( \frac{\alpha}{\tau_{Q}} \right)^{\alpha} + f(t)$, which however gives poorer results.