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Discrete-time modeling of woodwind instrument bores using wave variables

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A method for simulation of acoustical bores, useful in the context of sound synthesis by physical modeling of woodwind instruments, is presented. As with previously developed methods, such as digital waveguide modeling (DWM) [Smith, *Comput. Music J.* **16**, 74-91 (1992)] and the multi convolution algorithm (MCA) [Martínez *et al.*, *J. Acoust. Soc. Am.* **84**, 1620-1627 (1988)], the approach is based on a one-dimensional model of wave propagation in the bore. Both the DWM method and the MCA explicitly compute the transmission and reflection of wave variables that represent actual traveling pressure waves. The method presented in this report, the wave digital modeling (WDM) method, avoids the typical limitations associated with these methods by using a more general definition of the wave variables. An efficient and spatially modular discrete-time model is constructed from the digital representations of elemental bore units such as cylindrical sections, conical sections, and toneholes. Frequency-dependent phenomena, such as boundary losses, are approximated with digital filters. The stability of a simulation of a complete acoustic bore, which depends on the nature of the bore terminations, is investigated empirically. Results of the simulation of a full clarinet show that a very good concordance with classic transmission-line theory is obtained.

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1 Introduction

The objective of this paper is to present a method for time-domain simulation of woodwind instrument bores that can be used for musical sound synthesis purposes. The advantages of the time-domain approach over the frequency-domain approach have been discussed by Schumacher [1]; time-domain methods are inherently more suited to the prediction of perceptually important transient phenomena. The bore, assumed to behave linearly, is usually represented by its impulse response, and the interaction between the bore and the driver is modeled by a set of differential equations in combination with a convolution of the impulse response with the out-going pressure wave. A general framework for modeling of sustained musical tones on this basis was developed by McIntyre *et al.* [2].

However, musical use of a physical model of a woodwind instrument requires a spatially modular description of the bore. That is, the bore has to be modeled in such a way that the acoustic variables are defined at various specific points along the air column axis. This is because in order to vary the pitch, the player manipulates the acoustical response of the bore via opening and closing toneholes. A spatially modular description is also advantageous for the inclusion of non-linear effects in the bore; a recent experimental study on non-linear damping in woodwind toneholes [3] has indicated that these effects can have a significant influence on the internal and the radiated sound spectrum.

Two main techniques for obtaining a spatial description appear in the literature, namely digital waveguide modeling (DWM) [4, 5, 6] and the multi convolution algorithm (MCA) [7]. The latter has recently been adapted by Barjau *et al.* [8] for simulation of acoustical bores of arbitrary length. These techniques are remarkably similar in the sense that they are both based on a one-dimensional model of wave propagation in the bore, in which discontinuities in the bore are modeled through reflection and transmission of propagating waves; they differ mainly in the details of the numerical formulation. The main advantage of the DWM approach is that it allows the adjustment of the balance between accuracy and efficiency. Frequency-dependent phenomena (such as boundary or radiation losses) are modeled using a digital approximation of a continuous-domain formulation, where both the continuous formulation and the digital approximation technique may be chosen freely. In this respect the MCA approach is more limited, since it relies on the possibility of performing analytic inverse Fourier transforms of continuous-domain formulations. The possibility of adjusting the trade-off between accuracy and efficiency is particularly important with respect to musical sound synthesis; while efficiency is always an important criterion in this context [9], a synthesis method should anticipate and exploit the steady increasing in computational power of commonly available computing resources.

However, the feasibility of the DWM approach for modeling a variety of wind instrument bores has yet to be demon-

strated. This is partly due to numerical instability problems that can occur in simulations of conical sections [6]. Another obstacle is that the DWM approach does not provide methods for modeling of toneholes in a conical bore. This shortcoming stems directly from the fact that DWM techniques are specifically defined to simulate distributed systems; as a consequence, uncomputable loops are created when a conical section is directly connected to an acoustic unit with a non-zero instantaneous reflection (such as a tonehole).

This report presents a time-domain method, the wave digital modeling (WDM) method, that overcomes the majority of these problems. The method makes strong use of the classical analogy between electrical and acoustical systems, and combines DWM techniques with wave digital filter (WDF) techniques. The latter, which were originally designed for simulation of analog networks, are suited to modeling small acoustic units that may be considered as lumped elements, and are therefore conveniently complementary to DWM techniques. The main purpose of the present work is to formulate a unified description that encompasses both techniques, and that can be used as a general modeling framework for digital simulation of woodwind instrument bores.

The paper is organized as follows: the basic modeling principles are outlined in Sec. 2. Piecewise conical bore modeling has proved to be a useful concept in many previous studies on wind instrument modeling; how to apply WDM techniques for the simulation of such systems is discussed in Sec. 3, where accuracy is assessed via direct comparison with a transmission-line model, and stability properties are investigated empirically. In Sec. 4 a tonehole model is presented that can simulate holes of a wide variety of musically and physically feasible dimensions, and can be controlled dynamically. The main results are presented in Sec. 5, where the method is applied to the simulation of a full clarinet bore. Accuracy is again assessed via direct comparison with a transmission-line description. Finally the perspectives of this work are discussed in the conclusions.

2 General description of the method

WDF techniques are used for discretization of analog networks [10]. The resulting digital networks are called *wave digital filters* (WDFs). The classical analogy between electric and acoustic systems raises the possibility of employing WDF techniques for the discretization of lumped elements in a model of an acoustic system. WDF techniques are similar to DWM techniques in the sense that they both digitize continuous-time models using *wave variables*. As already suggested by Smith [11] and recently elaborated by Bilbao [12], a combined approach is possible. The latter work addresses the more general problem of numerically solving partial differential equations by means of methods that use wave variables. The methods presented in the present study are limited to cases in which lumped elements are modeled using WDF techniques and distributed elements are modeled using DWM techniques, and can be considered as a specific sub-class of the “wave” or “scattering” methods described by Bilbao. A rare musical acoustics application of such an

approach is the digital simulation of the force interaction between string and hammer in a piano, developed by Duyne and Smith [13]. The resulting piano hammer model is referred to as the *wave digital hammer*; an appropriate general term for the combined approach is *wave digital modeling*.

2.1 Modeling principles

The wave digital models presented in this work are derived from a transmission-line description of a woodwind bore. The approach thus requires that a transmission-line description or “equivalent network” is found for each individual bore component. In principle these networks may have any number of ports and may contain any type of linear element; the only restriction is that the complete network that represents the entire bore is stable.

The procedure for the derivation of the wave digital model of an individual bore component is similar to the derivation of a wave digital filter, and consists of three steps:

- (1) decomposition of the acoustic variables into wave variables.
- (2) discretization of frequency-dependent elements.
- (3) satisfaction of the computability condition.

Step (1) is accomplished by using the following relationships:

$$p_i = p_i^+ + p_i^- \quad \text{and} \quad U_i = \frac{p_i^+ - p_i^-}{R_i}, \quad (1)$$

where for port i , p_i is the pressure U_i is the volume velocity, while p_i^+ and p_i^- are the wave variables. The quantity R_i has the dimension resistance and, following WDF theory, is referred to as the *port-resistance*. In the case of a distributed acoustic element, the wave variables represent pressure-waves traveling through a certain medium. The port-resistance then equals the reference impedance that characterizes the medium; in the case of a wave traveling through an air-filled pipe, this is the characteristic impedance $Z_0 = \rho c/S$, where S is the cross-sectional pipe area, ρ is the mean air density, and c is the wave velocity. In the case of a lumped acoustic element, the wave variables do not represent waves that actually travel any distance; the decomposition is in this case merely a matter of mathematical description, and from an acoustical point of view the port-resistance may then be considered arbitrary. As in the derivation of WDFs, this freedom of choice is exploited to avoid delay-free loops in the final modeling structure. The decomposition of acoustic variables has to be carried out at each port of the system. Fig. 1 depicts a single port and its corresponding signal flow after decomposition.

Step (2) concerns the approximation in the digital domain of linear, frequency-dependent, continuous domain phenomena, which is realized in the present study by means of digital filters. A wave digital model contains various computational loops, in which these filters are placed. In order for the model to be stable, the gain of these loops must not exceed unity, and therefore the digital filters that are used here are designed such that their magnitude response is equal or less than unity at all frequencies. Three different types of filter

Figure 1: (a): A port and its associated acoustic variables. (b): The corresponding signal flow diagram after decomposition of the acoustic variables into wave variables.

Figure 2: A computational loop with transfer function $H(z)$ and port-resistance R .

design techniques are employed, each of them for a specific category of frequency-dependent elements. The first category comprises lumped circuit elements, such as inertances and compliances, which are mathematically described by a rational polynomial transfer function of the Laplace variable $s = j\omega$. As is customary in WDF theory, such elements are discretized via the bilinear transform (BT):

$$s = \beta \left(\frac{1 - z^{-1}}{1 + z^{-1}} \right), \quad (2)$$

where z^{-1} is the frequency-domain representation of a single delay of $T = 1/f_s$ seconds, $\beta = 2f_s$ is the bilinear operator, and f_s is the sample rate. Frequency-dependent phenomena which cannot be described with a rational polynomial transfer function fall into the second category. Digital approximation of such elements is predominantly carried out by means of infinite impulse response (IIR) filters. For the current purposes, the output-error minimization technique [14], which uses iterative gradient descent search methods to minimize a weighted least-square approximation error, is a particularly suitable method of IIR filter design, since it allows emphasis on accuracy at lower frequencies. Given that in wind instrument modeling, the frequencies below cut-off are of greater importance than the frequencies above cut-off, such “navigation” of the approximation error is helpful in improving the balance between accuracy and efficiency. The details of such IIR filter approximation can be found in previous work [6, 15]. The third category concerns the approximation of a fractional delay. As is explained in Sec. 2.3, fractional delay filters are required to simulate wave propagation in tubes of arbitrary length. In the present study, Lagrange FIR interpolation filters [5] are employed. The FIR and IIR filters used in the present study are typically of the order 3 to 5.

Step (3) is concerned with the computability of the resulting digital structure. Like a digital filter, a wave digital model is described mathematically by a system of difference equations. Such a system is called *computable* if the

Figure 3: (a): A side branch in a cylindrical pipe. (b): The equivalent network.

arithmetic operations prescribed by these equations can be ordered sequentially at each discrete-time instant [10]. In practice this condition is satisfied if the system contains no delay-free loops. In a wave digital model, such delay-free loops may arise in the discretization of a lumped element. One possible way to solve this problem is to insert a fictitious delay into the loop. However, such an approach leads to significant errors, unless a very high sample rate is used. Following WDF theory, these loops can be ensured to have at least one delay by choosing the appropriate port-resistance of that loop. For example, consider the loop in Fig. 2, in which $H(z)$ represents the digital transfer function of the loop. In a wave digital model, this transfer function can always be written in the form:

$$H(z) = \frac{b_0 + b_1 z^{-1} + \dots + b_N z^{-N}}{1 + a_1 z^{-1} + \dots + a_N z^{-N}}, \quad (3)$$

where the coefficients b_k, a_k depend on the port-resistance of the loop. The coefficient b_0 represents the factor by which the wave is multiplied before it is passed on *without delay*. Hence in order for this loop to be computable, this factor must be zero. If $H(z)$ represents a lumped element, the port-resistance may be chosen such that $b_0 = 0$.

2.2 Lumped elements

An acoustical device may be treated as a lumped acoustic element if the wavelength is considerably greater than its dimensions. An example of such a device is a short side branch in a cylindrical pipe (see Fig. 3a). In the low-frequency limit, the wavelength is long compared to the complicated flow patterns near the junction, and the acoustic behavior of the side branch is characterized by a simple shunt impedance network (see Fig. 3b). This is a three-port network to which Kirchhoff’s laws apply, i.e. :

$$p_1 = p_2 = p_3 \quad \text{and} \quad U_1 = U_2 + U_3. \quad (4)$$

The shunt impedance defines the relation between the acoustical pressure and volume velocity in the side branch:

$$p_3 = Z_s(\omega) U_3. \quad (5)$$

If inner length corrections are neglected, the side branch approximately acts as an inductance for an open branch (superscript ‘ (o) ’) and as a compliance for a closed branch (superscript ‘ (c) ’), i.e.

$$Z_s^{(o)}(\omega) = j\omega M \quad \text{and} \quad Z_s^{(c)}(\omega) = \frac{1}{j\omega C}, \quad (6)$$

where $M = (\rho t_e^{(o)})/S_h$ is the open-branch inductance, $C = S_h t_e^{(c)}/(\rho c^2)$ is the closed-branch compliance, S_h is the branch cross-section and $t_e^{(o)}$ and $t_e^{(c)}$ are the (real-valued) open-branch and closed-branch effective lengths, respectively.

A wave digital simulation of the open or closed branch is derived by carrying out the steps described in Sec. 2.1. Decomposition of the acoustic variables in Eqs. (4) yields the three-port junction scattering equations:

$$p_1^- = p_2^- + W, \quad (7)$$

$$p_2^+ = p_1^+ + W, \quad (8)$$

$$p_3^+ = p_1^+ + p_2^- + p_3^- + W, \quad (9)$$

where

$$W = k_1 [p_1^+ - p_3^-] + k_2 [p_2^- - p_3^-], \quad (10)$$

with the coefficients

$$k_1 = \frac{R_2 R_3 - R_1 R_3 - R_1 R_2}{R_2 R_3 + R_1 R_3 + R_1 R_2}, \quad (11)$$

$$k_2 = \frac{R_1 R_3 - R_2 R_3 - R_1 R_2}{R_2 R_3 + R_1 R_3 + R_1 R_2}. \quad (12)$$

Decomposition of to the acoustic variables in Eq. (5) gives:

$$p_3^- = R_s(\omega) p_3^+, \quad (13)$$

where $R_s(\omega)$ is a frequency-dependent wave reflectance:

$$R_s(\omega) = \frac{Z_s(\omega) - R_3}{Z_s(\omega) + R_3} \quad (14)$$

Note that this reflectance is not equivalent to what is usually referred to as a “plane wave reflectance”, since the wave variables p_3^+ and p_3^- do not represent actual traveling waves; from an acoustical point of view, their port-resistance value (R_3) may be chosen arbitrarily. Equation (14) is discretized using the BT, which yields the digitized wave reflectance or “wave digital reflectance”. In contrast to some other methods for discretizing analogue filters (such as the impulse invariance method), the use of the BT guarantees that there exists an R_3 for which the resulting digital filter transfer function has no instantaneous reflection. In the case of an open side branch, one obtains the wave digital reflectance:

$$R_s^{(o)}(z) = \frac{\alpha - z^{-1}}{1 - \alpha z^{-1}} \quad \text{with} \quad \alpha = \frac{\beta M - R_3}{\beta M + R_3}. \quad (15)$$

For a closed side branch, the discretization gives:

$$R_s^{(c)}(z) = \frac{\alpha + z^{-1}}{1 + \alpha z^{-1}} \quad \text{with} \quad \alpha = \frac{1 - \beta C R_3}{1 + \beta C R_3}. \quad (16)$$

Fig. 4 depicts the signal flow diagram of the obtained equations.

Figure 4: The signal flow diagram of a wave digital model of a shunt network. The 3-port junction block implements wave scattering. The wave digital reflectance $R_s(z)$ implements the shunt element.

The next step is to ensure computability of this structure. A possible delay-free loop exists in that p_3^+ depends on p_3^- according to (9), but p_3^- also depends on p_3^+ via (13). In order to avoid an uncomputable loop, the port-resistance R_3 must be chosen such that there is no instantaneous reflection from p_3^- into p_3^+ . In other words, the coefficient α in Eqs. (15) and (16) must equal zero. For the open branch, this is achieved by setting $R_3 = \beta M$, whereas for the closed branch the port-resistance must be set to $R_3 = 1/(\beta C)$. Substitution of these values into Eq. (16) yields

$$R_s^{(o)}(z) = -z^{-1} \quad \text{and} \quad R_s^{(c)}(z) = z^{-1}. \quad (17)$$

Thus the side branch is modeled with nothing more than a delay that is interfaced to the main bore with a three-port junction, where an extra sign-inversion is required when the branch is open.

2.3 Distributed elements

In the analogy between electrical circuits and tubular acoustic systems, distributed acoustic elements are modeled as transmission-lines. The simplest system of this type is one in which plane waves propagate through a cylindrical pipe. This system is mathematically described by a 2×2 matrix equation [16]:

$$\begin{bmatrix} p_1 \\ U_1 \end{bmatrix} = \begin{bmatrix} \cosh(\Gamma L) & Z_0 \sinh(\Gamma L) \\ Z_0^{-1} \sinh(\Gamma L) & \cosh(\Gamma L) \end{bmatrix} \begin{bmatrix} p_2 \\ U_2 \end{bmatrix}, \quad (18)$$

where p_1, U_1 and p_2, U_2 respectively denote the pressure and volume velocity at the input- and the output-end of the pipe, L is the pipe length, Γ is the complex propagation constant and Z_0 is the characteristic impedance. As with lumped acoustic elements, this description is only valid at low frequencies, this time because it takes into account only the principal mode of propagation. Again, the wave digital model is derived by carrying out the steps described in Sec. 2.1; in this case the procedure is in fact equivalent to applying DWM techniques. Since a pipe is a distributed system, the port-resistances on both sides of the system have to equal the characteristic impedance of the pipe, i.e. $R_1 = R_2 = Z_0$. The decomposition of the acoustic variables into wave variables then yields the relationships:

$$p_2^+ = [e^{-\Gamma L}] p_1^+ \quad , \quad p_1^- = [e^{-\Gamma L}] p_2^- \quad (19)$$

Figure 5: Signal flow diagram of a wave digital model of an open-ended pipe. Each z^{-N} unit indicates a delay-line. A *loss* unit represents a digital filter that approximates viscothermal losses, and an *FD* unit indicates a fractional delay filter. The open end is modeled with the reflectance filter $R_L(z)$.

If viscothermal losses are ignored, one may substitute $\Gamma = j\omega/c$, and the term $e^{-\Gamma L}$ is the frequency-domain description of a pure time delay L/c . As is customary in digital waveguide modeling, such a delay can be realized in discrete-time with a digital delay-line in cascade with a fractional delay filter [5]. The latter performs a numerical operation that is equivalent to time-interpolation, which allows modeling of wave propagation in a pipe of arbitrary length.

Viscothermal losses can be taken into account in the transmission-line description by using a lossy, frequency-dependent formulation of Γ , such as that described by Keefe [17], and can be assimilated into the wave digital model by cascading another digital filter with each delay-line. This filter is designed to approximate the continuous-domain expression:

$$H_{\text{loss}}(\omega) = e^{-(\Gamma - j\omega/c)L}. \quad (20)$$

The output-error method mentioned in section 2.1 can be used for this purpose. Frequency-dependent termination losses, such as occur at an open end of a pipe, are also approximated with a digital filter. This filter is designed to approximate the termination reflectance

$$R_L(\omega) = \frac{Z_L(\omega) - Z_0}{Z_L(\omega) + Z_0}, \quad (21)$$

where $Z_L(\omega)$ is the termination load. For most wind instruments, the radiation impedance of unflanged open ends as formulated by Levine and Schwinger [18] gives a suitable formulation of the termination load. Scavone [6] gives a convenient approximation to these formulas, that can be applied directly in the present context. After arranging the individual units in the appropriate way, the signal flow diagram of the wave digital model is obtained (see Fig. 5).

3 Piecewise conical bores

A woodwind bore may be considered as a succession of conical and cylindrical bore sections with a set of open or closed holes in their sides [7]. This section presents methods for simulation of piecewise conical bores with the WDM method.

3.1 Benade's equivalent network

A conical bore section is similar to a cylindrical bore section in the sense that both are pure waveguides. That is, plane

Figure 6: Equivalent circuit of a conical waveguide constructed of a uniform line, two shunt inertances, and transformer (after Benade [19]).

waves propagate without back-scattering in a cylindrical section, while spherical waves propagate in the same way in a conical section. This similarity is highlighted in the work of Benade [19], in which a conical section is represented as

an equivalent circuit consisting of a pair of inertances, a transformer, and a non-tapered duct that has the same length and small-end radius as the cone to be represented.

This equivalent network is depicted in Fig. 6. The values of the inertances are:

$$M_0 = \rho r_0 / S_0, \quad \text{and} \quad M_e = \rho r_e / S_e, \quad (22)$$

where r_0 and r_e are the distances from the cone apex and S_0 and S_e the wavefront areas at the left-hand and right-hand side of the cone, respectively. The distance from the cone apex is defined as negative if the apex is positioned on the right-hand side of the cone.

The methods presented in Sec. 2 can be used to derive a wave digital simulation of the equivalent circuit; a shunt inertance is modeled as a three-port junction with a single delay attached to one of its ports, and a bi-directional delay-line simulates the uniform line, with added filters for time-interpolation and inclusion of viscothermal losses. The transformer represents the decrease in pressure with increasing wavefront area, and can be modeled by adding a scaling factor to each delay-line. However, these scaling factors may be removed from the system without changing the overall reflectance at the input-end of the model. In such a scenario, one must apply them “extrinsically” when calculating the actual pressure at any point in the cone [15].

3.2 A junction of two conical sections

An equivalent network of two successive conical sections can be constructed by attaching two networks of the kind depicted in Fig. 6. A junction of two conical sections is thus described with a network which has the right-hand inertance $M_1 = -(\rho r_1)/S_1$ of the first cone in parallel with the left-hand inertance $M_2 = (\rho r_2)/S_2$ of the second cone. As pointed out in Ref. 19, this arrangement of the junction network is equivalent to a single shunt inertance

$$M_j = \frac{M_1 M_2}{M_1 + M_2} = \frac{\rho r_1 r_2}{r_1 S_2 - r_2 S_1}. \quad (23)$$

Figure 7: (a): The equivalent network of a conical junction that is terminated by an impedance load Z_L . (b): Signal flow diagram of a wave digital model. Each of the “line” units indicates a cascade of a digital delay-line, a fractional delay filter, and a loss filter. The “WD J” unit represents a wave digital junction, and the “WD-r J” unit represents a wave digital junction with zero instantaneous reflection in right-going direction.

It follows that the *wave digital junction* is derived in the same way as the wave digital structure of the open side branch discussed in Sec. 2.2. The signal flow of the wave digital junction is thus as depicted in Fig. 4. For certain junctions, the value of M_j is negative. The problems related to discrete-time modeling of such a negative inductance are discussed in section 3.5.

3.3 A junction of a conical section and a lumped element

Consider the network in Fig. 7a. This network is the electrical equivalent of a conical section that is terminated by a load $Z_L(\omega)$. This load could for example represent the open-end radiation impedance. The value of the shunt inductance at the termination is $M_j = -(\rho r)/S$, where r is the distance from the cone apex to the end of the cone, and S is the wave area at the cone end. Fig. 7b shows the wave digital model of this system. The term $R_L(z)$ indicates a digital filter approximation of the wave reflectance

$$R_L(\omega) = \frac{Z_L(\omega) - R_2}{Z_L(\omega) + R_2}, \quad (24)$$

where R_2 is the port-resistance as defined for the wave variables p_2^+ and p_2^- . If the taper junction modeled in discrete-time without taking special care concerning the wave reflectances (i.e. if R_2 is set equal to the local characteristic impedance), then the system would exhibit a delay-free loop, because in that scenario both the junction and the lumped element have a non-zero instantaneous reflection. In order to avoid such a delay-free loop, the junction near the lumped element is designed in such a way that has no immediate reflection of the wave incident from the right. It will therefore be referred to as a “WD- r junction”, where the letter r indicates that the junction has a zero instantaneous reflection in

right-going direction. The first few steps in the derivation of the WD- r junction are the same as in the case of the normal wave digital junction. That is, the basic structure is again a three-port junction (see Fig. 4), with an inductance wave reflectance

$$R_s(\omega) = \frac{j\omega M_j - R_3}{j\omega M_j + R_3} \quad (25)$$

attached to one of its ports. The port-admittance R_3 relates in the same way to the inductance as for the normal wave digital junction, i.e. $R_3 = \beta M_j$, and the wave digital reflectance again reduces to a negative delay. As explained in Sec. 2.2, Eqs. (7), (8), and (9) describe the scattering of wave variables at the three-port junction. In this case the port-resistance R_2 on the right-hand side may be chosen arbitrary because it connects directly to a lumped element. In order to derive a realizable structure, the instantaneous reflection of the incident wave p_2^- should be zero, which is true only if

$$R_2 = \frac{R_1 R_3}{R_1 + R_3}. \quad (26)$$

Substitution of Eq. (26) into Eqs. (11) and (12), yields the normal three-port equations (i.e. Eqs. (7), (8), and (9)), but with Eq. (10) replaced by:

$$W = -\left(\frac{R_1}{R_1 + R_3}\right) [p_1^+ - p_3^-]. \quad (27)$$

In the case of modeling a lumped element attached to the left side of the cone, a “WD- l junction” (that has a zero instantaneous reflection in the left-going direction) is required, and the derivation is again similar to that of the normal wave digital junction. In this case, the port-resistance R_1 has to be set to

$$R_1 = \frac{R_2 R_3}{R_2 + R_3}. \quad (28)$$

Substitution into Eqs. (11) and (12) again yields the normal three-port equations, but this time with Eq. (10) replaced by

$$W = -\left(\frac{R_2}{R_2 + R_3}\right) [p_1^+ - p_3^-]. \quad (29)$$

3.4 Example application

In this section the wave digital modeling approach is demonstrated for the bore configuration depicted in Fig. 8a. This configuration has been used in two previous studies [7, 8] on time-domain modeling. In this case, an interface that converts to the acoustic variables p and $Z_0 U$ is added to the input-end of the wave digital model (see Fig. 8b), so that the impulse response of the system is equivalent to the inverse Fourier transform of the input impedance (normalized by the characteristic impedance Z_0 of the cylindrical section). Third-order Lagrange interpolation filters were used for modeling fractional delay lengths and fourth-order IIR filters to approximate the viscothermal losses of each bore section. The wave digital reflectance $R_L(\omega)$ was approximated with a third-order IIR filter.

Figure 8: (a): Bore dimensions (in millimeter). (b): Signal flow diagram of the wave digital model.

For comparison, the impulse response was also computed by taking the inverse Fourier transform of a frequency-domain computation using transmission-line matrices, as described by Keefe [16]. Fig. 9 shows that the two resulting impulse responses are effectively identical.

3.5 Stability properties

For a normal wave digital junction (inertance given by Eq. (23)), the inertance remains positive for $S_2/r_2 > S_1/r_1$. Hence for certain, physically feasible junction configurations, the junction inertance M_j is negative. This amounts to using a negative port-resistance, which normally leads to an unstable filter structure [10]. Berners [20] has recently shown that the unstable junction filter is ultimately due to the presence of a trapped mode, such that traveling wave components due not constitute a complete basis set in the region surrounding the junction. However, this does not have to imply that the numerical formulation of a complete conical bore model is unstable, because such trapped modes do not exist for passively terminated systems [20].

In order to empirically test whether wave digital models of piecewise conical bore systems are stable, the bore configuration depicted in Fig. 10 was simulated. The simulation was run for a very long time (up to 250 seconds), using a variety of different input signals, and the resulting impulse response was analyzed; if the system is stable, than this signal must not exhibit amplitudal growth after the initial deflections. The simulation was run with various different terminations (i.e. anechoic, open, closed). The results show that the system remains unconditionally stable when viscothermal losses are not included. However when such losses are included, the system is in all cases unstable. When anechoic terminations are avoided, the signal starts to grow significantly only after a very long simulation time (typically 200 seconds); for such cases, the impulse response of the system, which typically has effectively decayed within less than one second for musical instrument bores, can be computed without any sig-

Figure 9: (a): Impulse response of the bore configuration depicted in Fig. 8a, computed as the inverse Fourier transform of the input impedance that was calculated with transmission-line matrices. (b): The same impulse response, computed with the wave digital model. These results may also be compared to those given in Ref. 7 (Fig. 8) and Ref. 8 (Fig. 6c).

Figure 10: (a): Bore configuration used in the stability tests. Dimensions are given in millimeter.

nificant error due to instability problems. It must be noted that these findings are subject to the precision of the floating points used in the computations; all results presented here are computed using 64-bit precision.

The fact that the inclusion of losses has a strong influence on the stability may be explained by the inconsistency in the formulation of the propagation constant. That is, the junctions are formulated using a *lossless* Γ , whereas wave propagation in the bore sections is formulated with a *lossy* Γ . Gilbert *et al.* [21] have proved that a continuous-domain formulation of a piecewise conical bore system is stable; this proof was carried out for a lossless as well as for a lossy version of the propagation constant. No such proof exists however for a continuous-domain formulation in which the propagation constant is inconsistent. It is worthwhile pointing out that not only the WDM method but also the MCA and the DWM approach are based on such an “inconsistent” continuous-domain formulation.

4 Toneholes

Physical modeling of woodwind instruments with application to sound synthesis requires a tonehole model that characterizes all tonehole states from open to closed. Scavone and Cook [22] developed a model that meets this requirement, but their “three-port tonehole model” has the limitation that the tonehole length is restricted to a minimum of $t = c/(2f_s)$. Hence an alternative method is required for simulation of shorter toneholes.

4.1 Lumped model

Following Dubos et al. [23], the shunt impedance of a tonehole is written

$$Z_s(\omega) = Z_h(\omega) + j\omega M_s, \quad (30)$$

where Z_h is the planar mode input impedance of the hole, and $M_s = t_s \rho / S_h$ is an inertance due to the higher order symmetrical modes at the intersection between the hole and the main bore, with

$$t_s = b \left(\frac{8}{3\pi i} - 0.193\delta - 1.09\delta^2 + 0.127\delta^3 - 0.71\delta^4 \right), \quad (31)$$

where $\delta = b/a$, b is the tonehole radius, and a is the main bore radius. At low frequencies, the open- and the closed-hole formulas for Z_h as given by Dubos et al. may be approximated with the lumped model formulas for Z_s in Eqs. (6), using the same effective length formulations as given in Ref. 23:

$$t_e^{(0)} = t_w + t_m + t_r, \quad t_e^{(c)} = t_w + t_m, \quad (32)$$

where t_w is the sum of the shortest geometrically tonehole length and t_m is the extra length due to the matching volume between the hole and the main bore:

$$t_m = \frac{b\delta}{8} (1 + 0.207\delta^3). \quad (33)$$

The term t_r in Eqs. (32) represents the length correction due to radiation, for which Nederveen [24] gives the empirical formula:

$$t_r = b [0.821 - 0.135(b/R_c) - 0.073(b/R_c)^4] + t_d, \quad (34)$$

where R_c is the outer radius of the hole, and t_d is the extra length correction for cases where a key of radius R_{pad} hanging a distance h above the hole ($t_d = 0$ for holes without key):

$$t_d = 0.613 b \left[(R_{pad}/b)^{0.18} (R_{pad}/h)^{0.39} - 1 \right]. \quad (35)$$

Fig. 11 compares the lumped model formulations of Z_h with the formulas by Dubos et al., for typical clarinet tonehole dimensions. In the lower frequency range, the approximation is extremely close.

An additional effect of inserting a hole in a woodwind bore is that the effective acoustic length of the bore is slightly reduced on both sides of the hole. In the wave digital modeling method, the total length-correction for an open or closed tonehole is formulated:

$$t_a = -b\delta^2 [2.72 + 0.540\delta + 0.285\delta^2]^{-1}. \quad (36)$$

Figure 11: Magnitude of the planar mode impedance Z_h of a sidehole of dimensions (in mm) $a = 7.5$, $b = 3.1$, $t_w = 7.0$, $R = 5.1$, $h = 2.4$. (a): Open hole. (b) Closed hole.

Thus if the lengths of the main bore sections on each side of the tonehole are l_1 and l_2 , they should be corrected to $l_1 + t_a/2$ and $l_2 + t_a/2$, respectively. This length-correction formulation approximates the series inertance formulation by Dubos et al.

4.2 Partially Open Holes

As discussed in section 2.2, the planar mode impedance Z_h of a sidehole is approximately that of a pure inertance or compliance depending on whether the hole is open or closed, respectively. For intermediate tonehole states (i.e. partially open holes), the tonehole volume V_b can be divided into an “open part” that behaves as an inertance, and a “closed part” that behaves as a compliance. These volumes operate in parallel, thus Eq. (30) becomes:

$$Z_s(\omega) = \frac{j\omega M}{1 - \omega^2 MC} + j\omega M_s, \quad (37)$$

where the compliance and inertance of the partially open hole are formulated:

$$C = (1 - g) \frac{S_h t_e^{(c)}}{\rho c^2} \quad \text{and} \quad M = \frac{\rho t_e^{(o)}}{g S_h}, \quad (38)$$

The parameter g expresses the tonehole state, defined as the ratio between *open* and *total* tonehole volume.

4.3 Discretization

The first step in the derivation of the wave digital tonehole model (i.e. the decomposition into wave variables) is analogous to the corresponding step discussed in Sec. 2.2. The

signal flow diagram is thus as depicted in Fig. 4. In this case, the wave digital reflectance is found after substitution of Eq. (37) into Eq. (14), and applying the BT, which yields:

$$R_s(z) = -\frac{\alpha_1 + \alpha_2 z^{-1} + \alpha_3 z^{-2} + z^{-3}}{1 + \alpha_3 z^{-1} + \alpha_2 z^{-2} + \alpha_1 z^{-3}}. \quad (39)$$

This is a digital all-pass filter with coefficients:

$$\alpha_1 = \frac{R_3(1 + \beta^2 MC) - \beta^3 M_s MC - \beta(M + Ms)}{R_3(1 + \beta^2 MC) + \beta^3 M_s MC + \beta(M + Ms)}, \quad (40)$$

$$\alpha_2 = \frac{R_3(3 - \beta^2 MC) + 3\beta^3 M_s MC - \beta(M + Ms)}{R_3(1 + \beta^2 MC) + \beta^3 M_s MC + \beta(M + Ms)}, \quad (41)$$

$$\alpha_3 = \frac{R_3(3 - \beta^2 MC) - 3\beta^3 M_s MC + \beta(M + Ms)}{R_3(1 + \beta^2 MC) + \beta^3 M_s MC + \beta(M + Ms)}. \quad (42)$$

In order to avoid a delay-free loop, R_3 must be chosen such that the wave p_3^+ entering $R_s(z)$ is not immediately reflected back towards the three-port scattering junction via p_3^- . This requires setting the filter coefficient $\alpha_1 = 0$, which means that the port-resistance must be set to

$$R_3 = \frac{\beta^3 M_s C + \beta(1 + M_s G)}{G + \beta^2 C}, \quad (43)$$

where the term $G = M^{-1}$ is introduced in order to allow the tonehole state parameter g in Eqs. (38) to go to zero, which corresponds to fully closing the tonehole. Substitution into Eq. (39) gives the final wave digital reflectance:

$$R_s(z) = -z^{-1} \left(\frac{\alpha_2 + \alpha_3 z^{-1} + z^{-2}}{1 + \alpha_3 z^{-1} + \alpha_2 z^{-2}} \right), \quad (44)$$

with

$$\alpha_2 = \frac{\beta^4 M_s C^2 + \beta^2 C(2M_s G - 1) + G(1 + M_s G)}{\beta^4 M_s C^2 + \beta^2 C(2M_s G + 1) + G(1 + M_s G)}, \quad (45)$$

$$\alpha_3 = \frac{-2\beta^4 M_s C^2 + 2G(1 + M_s G)}{\beta^4 M_s C^2 + \beta^2 C(2M_s G + 1) + G(1 + M_s G)} \quad (46)$$

4.4 A tonehole in a piecewise conical bore

As discussed in Sec. 3.1, a conical section may be modeled as a cylindrical waveguide in combination with two shunt inertances and an ideal transformer. A tonehole in a conical section may be considered as a system of two conical sections separated by a tonehole. The equivalent network of the hole and its connections to the two conical sections is depicted in Fig. 12a. This formulation does not require that the conical taper on the left side of the hole equals the taper on the right side. The wave digital model takes the form as depicted in Fig. 12b. In order to avoid delay-free loops, the cone inertances on the left and the right of the hole have to be modeled with a WD- r and a WD- l junction, respectively.

5 Results with a full clarinet bore

The bore of a clarinet can be divided into three parts; the mouthpiece, the main bore with toneholes, and the bell. The mouthpiece usually consists of a small entry section plus a

Figure 12: (a): Equivalent network of a partially open tonehole in a piecewise conical bore. (b): The wave digital model structure for digital simulation of this network. The “WD *hole*” unit indicates the wave digital tonehole model. The inertances associated with the conical sections on either side of the hole are modeled with a WD- r and a WD- l junction, such that no delay-free loops arise.

Figure 13: Cylinder-cone model of a Bundy mouthpiece. The conical section represents the tapered part of the mouthpiece that connects directly to the main bore, and the cylindrical section models the entry section closest to the reed channel. Dimensions are given in millimeter.

slightly tapered section that connects to the main bore. In most mouthpieces, there is a small cross-sectional step at the boundary between these two sections. In the low-frequency limit, the entry section may be modeled with a cylindrical section of equivalent volume. The tapered section can then be modeled as a perfect conical section, where the radius of the cylindrical section is chosen such that the cross-sectional step is the same as that in the real mouthpiece. Fig. 13 depicts such a “cylinder-cone model” of a *Bundy* mouthpiece.

The dimensions of the main bore and the toneholes of a Selmer clarinet (no. 1400) were measured. The geometrical data of the toneholes is given in Table 1. Each inter-hole section is modeled as cylindrical section of mean radius; because the main bore taper is extremely small, this simplification does not introduce any significant error. The remaining part of the bore (that is, the part starting directly after hole no. 1) is considered as the bell. The bell is typically mildly flared, and may therefore be approximated with a small number of piecewise conical sections.

Table 1: Dimensions of the Selmer clarinet. All dimensions are given in millimeters. x indicates the distance along the bore axis between the mouthpiece and the center of the hole.

hole no.	x	$2a$	$2b$	t_w	$2R_{pad}$	h
24	68.4	15.0	2.1	12.7	10.4	3.1
23	78.7	15.0	4.0	7.3	10.4	3.3
22	104.3	15.0	5.5	7.0	10.2	2.6
21	114.0	15.0	5.5	7.0	10.3	3.4
20	127.9	15.0	5.7	7.0	10.4	3.4
19	142.6	15.0	4.0	6.5	9.8	3.1
18	153.2	15.0	7.6	10.3	-	-
17	157.2	15.0	5.0	7.0	10.1	2.3
16	165.6	15.0	5.0	8.3	-	-
15	185.2	15.0	5.0	7.0	10.3	2.4
14	198.3	15.0	6.4	8.3	-	-
13	202.9	15.0	6.1	7.0	10.2	3.0
12	203.7	15.0	6.3	7.0	10.2	3.1
11	221.8	15.0	8.2	7.5	-	-
10	232.6	15.0	5.0	7.0	10.3	3.7
9	296.6	15.0	7.2	6.0	12.0	2.4
8	312.6	15.0	8.6	7.8	-	-
7	314.6	15.0	6.3	6.5	17.1	2.5
6	339.0	15.0	8.5	7.8	-	-
5	362.0	15.0	9.3	7.8	-	-
4	391.6	15.0	10.3	5.0	17.1	2.5
3	420.2	15.0	12.6	5.0	17.1	3.3
2	453.6	17.3	11.3	4.5	17.2	3.5
1	486.8	21.0	10.1	4.5	17.2	3.8

The choice of the sample rate of the wave digital model of the clarinet depends on the shortest inter-hole bore distance. In order to avoid the use of a very high sample rate, hole no. 13 was omitted from the model. The sample rate was then set to the smallest multiple of 44.1kHz that is sufficiently high for digital simulation of the shortest remaining inter-hole section, which is $f_s = 176.4\text{kHz}$. This sample rate choice allows for a simple conversion to a conventional audio rate, so that direct audio playback of output signals on standard audio equipment is possible.

In the model, fourth-order IIR filters were used for digital approximation of the boundary losses, and a fifth-order IIR filter was used to model the wave reflectance at the open-end. In order to avoid instability problems, viscothermal losses were not modelled in any of the conical sections.

Fig. 14a shows the input impedance of the complete clarinet bore as computed with two different transmission-line (TL) models. TL model (I) was formulated using the tonehole formulations of Dubos et al., while the lumped model described in section Sec. 4.1 was used in TL model (II). At the higher frequencies, the curves exhibit a slight difference in the position of the impedance peaks. This is mainly due to the different way in which the negative length-corrections associated with the toneholes are modeled. Fig. 14b compares TL model (II) with the wave digital model. As can be seen, the effects due to discretisation are very small. The main discrepancy is the higher amplitude of the first few resonance peaks, which is due to not taking into account the viscothermal losses in any of the conical sections.

Figure 14: Input impedance of the Selmer clarinet. Plot (a) compares the two transmission-line models; TL model (I) is computed using the tonehole formulations by Dubos et al., and TL model (II) is computed with the lumped model approximation to these formulations used in the present study. Plot (b) compares TL model (II) to the wave digital model. The computations were done with the fingering for note F_3 (all holes closed except no. 1 and no. 3).

6 Conclusions

Woodwind instrument bores can be accurately simulated in discrete time with the WDM method. Using a transmission-line description of the bore, a spatially modular bore model is constructed from which a traveling-wave based simulation is derived through decomposition into wave variables and discretization of the frequency-dependent elements. Toneholes are modeled in such a way that each possible state (from open to closed) can be simulated and controlled dynamically. The possibility of such parameterization is essential in the context of generating musically meaningful model output with the model. Another key-feature of the WDM method is the flexibility concerning the formulation and digital approximation of frequency-dependent phenomena, which allows adjustment of the trade-off between accuracy and efficiency. As such, wave digital modeling is well suited to synthesis-orientated simulation of wind instruments.

Although the present study is limited to calculation of vibrations inside the bore of a woodwind instrument, the WDM method can be adapted to predict the sound radiated from the instrument, and also to simulate of other wind instru-

ments, such as brass instruments and flutes.

The results presented in Sec. 3.4 and Sec. 5 show that a wave digital model closely approximates the transmission-line description of an acoustic bore. The main discrepancies were found to be due to the following limitations: (1) the accuracy is sensitive to the way in which the length-corrections associated with the toneholes are taken into account, and (2) the simulation of a piecewise conical bore system is stable only if viscothermal losses are not taken into account in any of the conical sections. Limitation (2) has no significant consequences for simulation of bores which are predominantly cylindrical, such as the clarinet. However, for simulation of conical bore instrument, such as the saxophone, further research is required. In particular, stable simulation of lossy conical bores would require a discrete-time junction model that is formulated using a lossy version of the propagation constant. The effects of limitation (1) are very small at low frequencies. Effects due to other discretization aspects and further simplifications are usually extremely small; the results show that if the transmission-line model of a full clarinet bore is adapted such that the effects of limitation (1) are removed from the comparisons, the models no longer show any significant discrepancy in the important frequency range. Possibly limitation (1) can be removed in future work, by modeling the length corrections with wave digital modules that simulate negative series inertances.

A further useful improvement could be include some form of resistance in the wave digital tonehole model. Nederveen [24] includes a non-linear resistive term in the tonehole shunt impedance. Such a resistive term could explain particular phenomena observed with very small holes, and may need to be included for accurate simulation of register holes.

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