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Nonlinear excitations in electron-positron-ion plasmas in accretion disks of active galactic nuclei

W. M. Moslem^{a)}

Institut für Theoretische Physik IV, Fakultät für Physik und Astronomie, Ruhr-Universität Bochum, D-44780 Bochum, Germany and Department of Physics, Faculty of Education-Port Said, Suez Canal University, Egypt

I. Kourakis^{b)}

Centre for Plasma Physics (CPP), Department of Physics and Astronomy, Queen's University Belfast, BT7 1 NN, Northern Ireland and Institut für Theoretische Physik IV, Fakultät für Physik und Astronomie, Ruhr-Universität Bochum, D-44780 Bochum, Germany

P. K. Shukla^{c)}

Institut für Theoretische Physik IV, Fakultät für Physik und Astronomie, Ruhr-Universität Bochum, D-44780 Bochum, Germany, School of Physics, University of Kwazulu-Durban, Durban 4000, South Africa, Department of Physics, Umeå University, SE-90187 Umeå, Sweden, Max-Planck Institut für Extraterrestrische Physik, D-85741 Garching, Germany, GoLP/Instituto Superior Técnico, 1049-001 Lisbon, Portugal, CCLRC Centre for Fundamental Physics, Rutherford Appleton Laboratory, Chilton, Didcot, Oxon OX11 0QX, United Kingdom, and SUPA Department of Physics, University of Strathclyde, Glasgow G 40NG, United Kingdom

R. Schlickeiser^{d)}

Institut für Theoretische Physik IV, Fakultät für Physik und Astronomie, Ruhr-Universität Bochum, D-44780 Bochum, Germany

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The propagation of acoustic nonlinear excitations in an electron-positron-ion (e-p-i) plasma composed of warm electrons and positrons, as well as hot ions, has been investigated by adopting a two-dimensional cylindrical geometry. The electrons and positrons are modeled by hydrodynamic fluid equations, while the ions are assumed to follow a temperature-parametrized Boltzmann distribution (the fixed ion model is recovered in the appropriate limit). This situation applies in the accretion disk near a black hole in active galactic nuclei, where the ion temperature may be as high as 3 to 300 times that of the electrons. Using a reductive perturbation technique, a cylindrical Kadomtsev-Petviashvili equation is derived and its exact soliton solutions are presented. Furthermore, real situations in which the strength of the nonlinearity may be weak are considered, so that higher-order nonlinearity plays an important role. Accordingly, an extended cylindrical Kadomtsev-Petviashvili equation is derived, which admits both soliton and double-layer solutions. The characteristics of the nonlinear excitations obtained are investigated in detail. © 2007 American Institute of Physics. [DOI: 10.1063/1.2795127]

I. INTRODUCTION

In contrast to ordinary plasmas consisting of electrons and positive ions, linear and nonlinear waves in electron-positron (e-p) plasmas behave differently.¹ Such plasmas are believed to exist in the early universe,² in active galactic nuclei,³ in pulsar magnetospheres,⁴ in solar atmosphere,⁵ at the center of our own galaxy,⁶ and are also of relevance in inertial confinement fusion schemes using ultraintense lasers.⁷ A number of pioneering experiments⁸ have been carried out to produce positrons in laboratory. Recently, Helander and Ward⁹ have discussed the possibility of e-p production in large tokamaks due to collisions between multi-MeV runaway electrons and thermal particles.

An electron-positron plasma usually behaves as a fully

ionized gas consisting of electrons and positrons, as seen in the solar atmosphere.⁵ Since many of the astrophysical plasmas contain ions besides the electrons and positrons, it is important to study the behavior of nonlinear wave motions in an electron-positron-ion (e-p-i) plasma. During the past three decades, e-p and e-p-i plasmas have attracted significant attention among researchers.¹⁰⁻¹⁸ Linear and nonlinear wave propagation in e-p and e-p-i plasmas have been studied by using different models. For example, Popel *et al.*¹³ investigated ion-acoustic solitons in three-component plasmas, whose constituents are electrons, positrons, and singly charged ions. It is found that the presence of the positron component can result in the reduction of the ion-acoustic amplitude. Nejoh¹⁵ studied the effect of the ion temperature on large-amplitude ion-acoustic waves in an e-p-i plasma. He found that the ion temperature increases the maximum Mach number and decreases the amplitude of the ion-acoustic waves. Also, the region of the existence of the ion-acoustic waves spreads as the ion temperature decreases.

^{a)}Electronic mail: wmm@tp4.rub.de; wmmoslem@hotmail.com

^{b)}Electronic mail: ioannis@tp4.rub.de; i.kourakis@qub.ac.uk

^{c)}Electronic mail: ps@tp4.rub.de

^{d)}Electronic mail: rsch@tp4.rub.de

Shukla and Shukla¹⁷ have shown that magnetic field aligned equilibrium relativistic plasma flows can excite electromagnetic fluctuations in a magnetized e - p plasma. The dispersion relation admits purely growing instabilities of electromagnetic perturbations across the ambient magnetic field direction.

Relying on physical principles, the ionic temperature has tacitly been assumed to be lower than the electron and positron temperatures, and hence the ions were treated as a fluid, while the electrons and positrons were treated either as a fluid or as Boltzmann-like. This assumption is valid for many astrophysical situations and laboratory experiments. However, it is reported that the inner region of the accretion disk in active galactic nuclei¹⁹ bears electron temperatures of 10^9 K and ion temperatures 3 to 300 times hotter than that of the electrons. Our purpose in this paper is to investigate the behavior of the electrostatic waves in the inner region of the accretion disk by adopting an appropriate model. We consider a three-component e - p - i plasma in which the electrons and positrons are modeled by hydrodynamic (plasma-fluid) equations and the ions are assumed to follow a Boltzmann distribution. We shall examine the effects of the electron-to-ion temperature ratio, the positron-to-electron density ratio, and the soliton propagation speed on the properties of nonlinear electrostatic excitations (solitons and double layers).

In Sec. II, we present the governing equations for the electrostatic waves in our e - p - i plasma. In Sec. III, the reductive perturbation method is employed to derive a cylindrical Kadomtsev-Petviashvili equation (CKP). In Sec. IV, we obtain an appropriate equation that describes the evolution of the nonlinear ion-acoustic waves, including higher-order nonlinearity. The possibility of propagating electrostatic solitons and double layers is discussed in Sec. V. Finally, our results are summarized in Sec. VI.

II. BASIC EQUATIONS

We consider a collisionless, unmagnetized three-component plasma consisting of the electrons, positrons, and singly charged positive ions. The nonlinear propagation of electrostatic waves in a two-dimensional (2D) cylindrical geometry is governed by the electron (positron) [denoted by the index e (p)] continuity equation(s)

$$\frac{\partial n_{e,p}}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r n_{e,p} u_{e,p}) + \frac{1}{r} \frac{\partial}{\partial \theta} (n_{e,p} v_{e,p}) = 0, \quad (1)$$

and the e (p) momentum equation(s),

$$\frac{\partial u_{e,p}}{\partial t} + u_{e,p} \frac{\partial u_{e,p}}{\partial r} + \frac{v_{e,p}}{r} \frac{\partial u_{e,p}}{\partial \theta} - \frac{v_{e,p}^2}{r} + 2\sigma_{e,p} \frac{\partial n_{e,p}}{\partial r} + q_{e,p} \frac{\partial \varphi}{\partial r} = 0, \quad (2)$$

and

$$\frac{\partial v_{e,p}}{\partial t} + u_{e,p} \frac{\partial v_{e,p}}{\partial r} + \frac{v_{e,p}}{r} \frac{\partial v_{e,p}}{\partial \theta} + \frac{u_{e,p} v_{e,p}}{r} + 2 \frac{\sigma_{e,p}}{r} \frac{\partial n_{e,p}}{\partial r} + \frac{q_{e,p}}{r} \frac{\partial \varphi}{\partial \theta} = 0. \quad (3)$$

Equations (1)–(3) are closed with the help of the Poisson equation,

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \varphi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \varphi}{\partial \theta^2} + \sum_{e,p} q_{e,p} n_{e,p} + n_i = 0. \quad (4)$$

We shall assume that the ions are Boltzmann-distributed and their ion number density is

$$n_i = \beta \exp(-\sigma_i \varphi). \quad (5)$$

This choice, though seemingly counterintuitive, is motivated by accretion disk plasma gas models,¹⁹ where the ions are reported to be up to 300 times hotter than the electrons (a more conventional fixed-ion hypothesis may also be adopted upon simply setting $\sigma_i=0$, viz., $n_i=\text{const}$, in the forthcoming algebra). In Eqs. (1)–(5), r and θ are the radial and polar angle coordinates, n_e (p) and n_i are the number densities of electrons (positrons) and ions, respectively, u_e (p) and v_e (p) are the electron (positron) velocities in the r and θ directions, respectively, and φ is the electrostatic potential. The time and space variables, t and r , are normalized by the electron plasma period $\omega_{pe}^{-1} [= (4\pi e^2 n_{e0}/m)^{-1/2}]$ and the electron Debye radius $\lambda_{De} [= (T_e/4\pi e^2 n_{e0})^{1/2}]$, respectively. The densities ($n_{e/p/i}$), velocities ($u_{e/p}$, $v_{e/p}$), and electrostatic potential φ are normalized by the unperturbed equilibrium number electron density n_{e0} , the electron thermal speed $C_{de} [= (T_e/m)^{1/2}]$, and T_e/e , respectively (Boltzmann's constant is omitted everywhere). We have denoted $\sigma_e=1$ for the electrons, $\sigma_p=T_p/T_e$ for the positrons, and $\sigma_i=T_e/T_i$ for the positive ions. Finally, we have defined the dimensionless parameters $q_e=-1$ (for the electrons) and $q_p=1$ (for the positrons).

The quasineutrality condition at equilibrium reads

$$\delta + \beta = 1, \quad (6)$$

where $\beta=n_{i0}/n_{e0}$ and $\delta=n_{p0}/n_{e0}$ denote the ion-to-electron and positron-to-electron unperturbed density ratio(s).

III. QUASICONCENTRIC NONLINEAR WAVE PROPAGATION—DERIVATION OF THE CYLINDRICAL KADOMTSEV-PETVIASHVILI EQUATION

In view of investigating the nonlinear propagation of the electrostatic waves, we shall employ a reductive perturbation technique.^{20,21} We anticipate excitations with nearly cylindrical symmetry, so the angle subtended by the wavefront in the nearly concentric case may be assumed to be small. The independent variables can be stretched as

$$\xi = \varepsilon^{1/2}(r - \lambda t), \quad \eta = \varepsilon^{-1/2}\theta, \quad \text{and} \quad \tau = \varepsilon^{3/2}t, \quad (7)$$

where ε is a small parameter and λ is the wave propagation speed, to be determined later. The dependent variables are expanded as

$$n_{e,p} = \alpha + \varepsilon n_{e,p}^{(1)} + \varepsilon^2 n_{e,p}^{(2)} + \varepsilon^3 n_{e,p}^{(3)} + \cdots, \quad (8)$$

$$u_{e,p} = \varepsilon u_{e,p}^{(1)} + \varepsilon^2 u_{e,p}^{(2)} + \varepsilon^3 u_{e,p}^{(3)} + \cdots, \quad (9)$$

$$v_{e,p} = \varepsilon^{3/2} v_{e,p}^{(1)} + \varepsilon^{5/2} v_{e,p}^{(2)} + \varepsilon^{7/2} v_{e,p}^{(3)} + \cdots, \quad (10)$$

$$\varphi = \varepsilon \varphi^{(1)} + \varepsilon^2 \varphi^{(2)} + \varepsilon^3 \varphi^{(3)} + \cdots, \quad (11)$$

where $\alpha=1$ for the electrons and $\alpha=\delta$ for the positrons.

The perturbation method to be adopted here consists in substituting Eqs. (7)–(11) into the model Eqs. (1)–(5), and isolating, and then solving, distinct orders in ε . The lowest order in ε gives

$$n_{e,p}^{(1)} = \frac{q_{e,p}\alpha}{S_{e,p}}\varphi^{(1)}, \quad u_{e,p}^{(1)} = \frac{q_{e,p}\lambda}{S_{e,p}}\varphi^{(1)}, \quad \frac{\partial v_{e,p}^{(1)}}{\partial \xi} = \frac{q_{e,p}}{\tau S_{e,p}} \frac{\partial \varphi^{(1)}}{\partial \eta}, \quad (12)$$

where $S_{e,p} = \lambda^2 - 2\sigma_{e,p}\alpha$, i.e., $S_e = \lambda^2 - 2$ and $S_p = \lambda^2 - 2\sigma_p\delta$. The sign of the latter may be positive or negative, depending on the values of the plasma parameters λ , $\sigma_{e,p}$ and α . Note that the (leading-order) perturbation of the electric potential determines the corresponding variation of the electron and positron density and velocity. For instance, physically speaking, an electron density compression may usually be expected where a positron rarefaction is found, and vice versa; note, however, that this is not always the case, since $n_e^{(1)} = -1/S_e\varphi^{(1)}$ and $n_p^{(1)} = \delta/S_p\varphi^{(1)}$ bear the same sign between $\sqrt{2}$ and $\sqrt{2}\sigma_p\delta$ (opposite signs elsewhere).

The Poisson equation provides the compatibility condition

$$\frac{1}{\lambda^2 - 2} + \frac{\delta}{\lambda^2 - 2\sigma_p\delta} = \beta\sigma_i. \quad (13)$$

The next order in the perturbation yields

$$\lambda \frac{\partial n_{e,p}^{(1)}}{\partial \tau} - \lambda^2 \frac{\partial n_{e,p}^{(2)}}{\partial \xi} + \lambda \alpha \frac{\partial u_{e,p}^{(2)}}{\partial \xi} + \lambda \frac{\partial}{\partial \xi} (n_{e,p}^{(1)} u_{e,p}^{(1)}) + \frac{\alpha}{\tau} u_{e,p}^{(1)} + \frac{\alpha}{\tau} \frac{\partial v_{e,p}^{(1)}}{\partial \eta} = 0, \quad (14)$$

$$\frac{\partial u_{e,p}^{(1)}}{\partial \tau} - \lambda \frac{\partial u_{e,p}^{(2)}}{\partial \xi} + u_{e,p}^{(1)} \frac{\partial n_{e,p}^{(1)}}{\partial \xi} + 2\sigma_{e,p} \frac{\partial n_{e,p}^{(2)}}{\partial \xi} + q_{e,p} \frac{\partial \varphi^{(2)}}{\partial \xi} = 0, \quad (15)$$

and

$$\frac{\partial^2 \varphi^{(1)}}{\partial \xi^2} + \sum_{e,p} q_{e,p} n_{e,p}^{(2)} - \beta\sigma_i \varphi^{(2)} + \frac{1}{2} \beta\sigma_i^2 \varphi^{(1)2} = 0. \quad (16)$$

Eliminating the second-order perturbed quantities and making use of the first-order results, we obtain a nonlinear partial-derivative equation (PDE) in the form of the cylindrical Kadomtsev-Petviashvili (CKP) equation,

$$\frac{\partial}{\partial \xi} \left(\frac{\partial \varphi^{(1)}}{\partial \tau} + A \varphi^{(1)} \frac{\partial \varphi^{(1)}}{\partial \xi} + B \frac{\partial^3 \varphi^{(1)}}{\partial \xi^3} + \frac{1}{2\tau} \varphi^{(1)} \right) + \frac{1}{2\lambda\tau^2} \frac{\partial^2 \varphi^{(1)}}{\partial \eta^2} = 0. \quad (17)$$

The nonlinearity and diffusion coefficients, A and B , respectively, read

$$A = -\frac{3\lambda}{2} \left(\frac{1}{\lambda^2 - 2} - \frac{1}{\lambda^2 - 2\sigma_p\delta} \right) + \beta\sigma_i^2, \quad (18)$$

$$B = \frac{(\lambda^2 - 2)^2}{2\lambda} + \frac{(\lambda^2 - 2\sigma_p\delta)^2}{2\lambda\delta}. \quad (19)$$

We note that B is usually greater than zero, while the sign of A depends on specific parameter values, as we shall see below. Note that, combining with Eq. (13), the dependence of A and B on σ_p is eliminated. So, A and B depend on λ , σ_i and β only.

To obtain an exact solitary wave solution of Eq. (17), we shall use the following substitution:²¹

$$\zeta = \xi - \frac{\lambda}{2} \eta^2 \tau \quad \text{and} \quad \varphi^{(1)} = \varphi^{(1)}(\zeta, \tau), \quad (20)$$

into Eq. (17). We thus obtain the standard Korteweg-de Vries (KdV) equation

$$\frac{\partial \varphi^{(1)}}{\partial \tau} + A \varphi^{(1)} \frac{\partial \varphi^{(1)}}{\partial \zeta} + B \frac{\partial^3 \varphi^{(1)}}{\partial \zeta^3} = 0. \quad (21)$$

In order to obtain a steady-state traveling-wave solution of Eq. (21), we introduce the transformation

$$\chi = \zeta - U_0 \tau, \quad (22)$$

where U_0 is a real constant representing the speed of the moving wavefront. Assuming the boundary conditions $\varphi \rightarrow 0$ and $d\varphi/d\chi \rightarrow 0$ at $|\chi| \rightarrow \infty$, we obtain a solitary wave solution (KdV soliton),

$$\varphi^{(1)} = \varphi_0 \operatorname{sech}^2 \left(\frac{\chi - U_0 \tau}{W} \right), \quad (23)$$

where $\varphi_0 = 3U_0/A$ is the maximum (potential perturbation) amplitude and $W = \sqrt{4B/U_0}$ measures the spatial extension (width) of the localized pulse. We note that $\varphi_0 W^2 = 12B/A = \text{const}$ (for given λ), in agreement with the standard KdV result.²² However, the dependence of φ_0 and W on λ is more complex, as we shall see below. Combining Eq. (20) with Eq. (23), we obtain the bell-shaped potential excitation

$$\varphi^{(1)} = \varphi_0 \operatorname{sech}^2 \left(\frac{\xi - \left(U_0 + \frac{\lambda}{2} \eta^2 \right) \tau}{W} \right). \quad (24)$$

It is clear that the propagation speed of the solitary wave is angle-dependent, i.e., the cylindrical wave will slightly deform as time or propagation polar angle progress.

Note that the sign of the nonlinear coefficient A affects the sign of φ_0 , which in turn determines the sign of the potential perturbation $\varphi^{(1)}$; therefore, both positive and negative pulses may possibly be obtained, for different parameter values, respectively, representing a potential hump or a dip (an electrostatic potential hole). Furthermore, note that the soliton amplitude φ_0 is proportional to the soliton speed U_0 and inversely proportional to the soliton width W , hence faster KdV soliton solutions will be taller and narrower, while slower ones will be shorter and wider. The qualitative features of the KdV solitons are thus recovered.²²

Relying on the above findings, we shall now investigate the effect of relevant physical quantities, namely the electron-to-ion temperature ratio (σ_i), the positron-to-electron density ratio (δ), and the soliton propagation speed

FIG. 1. (a) The soliton amplitude ϕ_0 is depicted against the propagation speed λ for $\sigma_i=1/300$, $\delta=0.4$ (solid curve), $\delta=0.9$ (dashed curve), and $\sigma_i=1/3$ and $\delta=0.4$ (dotted curve). (b) The soliton width W is depicted against λ for $\delta=0.1$ (solid curve), $\delta=0.5$ (dashed curve), and $\delta=0.99$ (dotted curve). Here we have used $U_0=0.1$.

(λ) on the properties of nonlinear electrostatic excitations in e - p - i plasmas. See that σ_p is prescribed from Eq. (13), once the remaining parameters are prescribed.

Figure 1(a) depicts the soliton amplitude as a function of the propagation speed λ . The pulse will have a bell-shaped form for a low propagation speed, representing a localized potential increase (a positive excitation, viz. $\phi_0 > 0$, i.e., a hump). An inverted bell-shaped form (a negative perturbation, viz. $\phi_0 < 0$, i.e., an electrostatic hole, a potential void) occurs for higher values of λ . The positive (pulse) soliton profile becomes less sharp by increasing λ up to $\sqrt{2} \approx 1.414$. In fact, A has a pole at $\lambda = \sqrt{2}$; nonlinearity becomes infinitely high in the neighborhood, so no dispersion-nonlinearity-managed excitations are expected to occur in the vicinity of this value. On the other hand, the negative (pulse) excitation profile becomes sharper (higher) by increasing λ . Reducing the ionic temperature, i.e., increasing σ_i , leads to decreases (increases) in the amplitude of positive (negative) solitons. Furthermore, increasing the value of δ (i.e., increasing the positron concentration, or decreasing the ions) makes the soliton profile sharper, i.e., it increases the value of ϕ_0 . Figure 1(b) depicts the effect of the propagation speed λ and the positron-to-electron density ratio δ on the soliton width, as obtained from the relation $W = \sqrt{4B/U_0}$. It is clear that the positive (pulse) soliton width decreases with λ , while the opposite happens to negative (hole) solitons. Fur-

FIG. 2. (Color online) The electrostatic potential $\phi^{(1)}$ is depicted against the radial and angular coordinates, r and θ , respectively: (a) compressive solitary pulse for $\lambda=1$ and (b) rarefactive solitary pulse for $\lambda=2$. The other plasma parameters are $\sigma_i=1/300$, $\delta=0.4$, $U_0=0.1$, and $t=1$.

thermore, the soliton width becomes wider by increasing the value of δ . The electrostatic potential pulse excitations $\phi^{(1)}$ are depicted in Fig. 2, against the radial coordinate r and the polar angle θ . It is observed that the propagation of both positive and negative soliton pulses in the radial direction is slightly distorted and accelerated due to (weak) angular, i.e., nongyrotropic effects. Note that the real value of the electrostatic potential is $(T_e/e)\phi^{(1)} \approx 86.2\phi^{(1)}$ (in KV), for an electron temperature of 10^9 K.¹⁹

IV. EXCITATIONS AT CRITICALLY NONLINEAR PLASMA COMPOSITIONS

Summarizing the above analysis, we have developed a theory for nonlinear excitations in e - p - i plasmas, in fact reducing the system of plasma-fluid equations to a CKP equation. An appropriate traveling-wave ansatz has enabled us to reduce the CKP equation further to a KdV equation, which features a nonlinearity coefficient A given by Eq. (18). The solutions thus obtained are sustained by a balance between dispersion (expressed by the coefficient B) and nonlinearity (expressed by A). An important question arises here regarding the strength of the nonlinearity. Specifically, one may wonder whether the coefficient A may acquire small values, and what happens then. Interestingly, this appears to be the

case in accretion disks. For instance, adopting values from Ref. 19 for $\sigma_i=1/3$, $\delta=0.1$, and $\lambda=2.38$, we obtain $A=0.1$. Therefore, we may anticipate that higher-order nonlinearity enters into play, if A is of the order of, say, ε . To answer this question, one should obtain an appropriate equation that describes the evolution of the system in this case. In this section, we shall therefore explicitly assume that $A \sim \varepsilon$, in order to examine the behavior of the nonlinear waves in an e - p - i plasma at critical compositions. A similar discussion has been carried out in multispecies e - i plasmas; see, e.g., Ref. 23.

Adopting a slightly different version of the reductive perturbation technique, we now introduce stretched space-time coordinates

$$\xi = \varepsilon(r - \lambda t), \quad \eta = \varepsilon^{-1}\theta, \quad \text{and} \quad \tau = \varepsilon^3 t. \quad (25)$$

The dependent variables are expanded in a manner similar to Eqs. (8)–(11). Using the new stretching Eq. (25) along with Eqs. (8)–(11) into Eqs. (1)–(5) and equating the terms arising at the lowest order in ε , we obtain the same linearized expressions (12) and the compatibility relation (13).

The next order in ε gives

$$n_{e,p}^{(2)} = \frac{3\lambda^2\alpha}{2S_{e,p}^3}\varphi^{(1)2} + \frac{q_{e,p}\alpha}{S_{e,p}}\varphi^{(2)}, \quad (26)$$

$$u_{e,p}^{(2)} = \frac{1}{2S_{e,p}^3}(\lambda^3 + 4\sigma_{e,p}\lambda\alpha)\varphi^{(1)2} + \frac{q_{e,p}\lambda}{S_{e,p}}\varphi^{(2)}. \quad (27)$$

The next higher order, $O(\varepsilon^3)$ of the Poisson equation, upon using the first- and second-order solutions, leads to the extended CKP (ECKP) equation,

$$\begin{aligned} \frac{\partial}{\partial \xi} \left(\frac{\partial \varphi^{(1)}}{\partial \tau} + A\varphi^{(1)} \frac{\partial \varphi^{(1)}}{\partial \xi} + C\varphi^{(1)2} \frac{\partial \varphi^{(1)}}{\partial \xi} + B \frac{\partial^3 \varphi^{(1)}}{\partial \xi^3} + \frac{1}{2\tau} \varphi^{(1)} \right) \\ + \frac{1}{2\lambda\tau^2} \frac{\partial^2 \varphi^{(1)}}{\partial \eta^2} = 0, \end{aligned} \quad (28)$$

where A and B are given in Eqs. (18) and (19), and

$$C = \frac{3\lambda}{4} \left(\frac{5\lambda^2 + 8}{\lambda^2 - 2} + \frac{5\lambda^2 + 8\sigma_p\delta}{\lambda^2 - 2\sigma_p\delta} \right) - \frac{1}{2}\beta\sigma_i^3. \quad (29)$$

By applying the ansatz (20) in the evolution Eq. (28), we obtain the extended KdV (EKdV) equation,²⁴

$$\frac{\partial \varphi^{(1)}}{\partial \tau} + \frac{A}{2} \frac{\partial \varphi^{(1)2}}{\partial \xi} + \frac{C}{3} \frac{\partial \varphi^{(1)3}}{\partial \xi} + B \frac{\partial^3 \varphi^{(1)}}{\partial \xi^3} = 0. \quad (30)$$

V. SOLUTION OF THE EKDV EQUATION VIA A PSEUDOPOTENTIAL METHOD

The EKdV equation (30) may be solved by quadrature. Using the traveling-wave transformation (22), and integrating, by using vanishing boundary conditions at infinity, we obtain

$$\frac{1}{2} \left(\frac{d\varphi^{(1)}}{d\chi} \right)^2 + V(\varphi^{(1)}) = 0. \quad (31)$$

This relation suggests that the evolution of a solitary excitation is analogous to the problem of motion of a unit mass in a (Sagdeev-like²⁵) pseudopotential, given by

$$V(\varphi^{(1)}) = -\frac{U_0}{2B}\varphi^{(1)2} + \frac{A}{6B}\varphi^{(1)3} + \frac{C}{12B}\varphi^{(1)4}. \quad (32)$$

In the following, we shall investigate the form and properties of the solutions related to the above pseudo-energy-balance equation.

A. Solitary-wave solution

The solitary-wave solution(s) of Eq. (31) may be cast in the form

$$\begin{aligned} \left(\frac{d\varphi^{(1)}}{d\chi} \right)^2 &= -\frac{C}{6B}\varphi^{(1)2} \left(\varphi^{(1)2} + \frac{2A}{C}\varphi^{(1)} - \frac{6U_0}{C} \right) \\ &= -\frac{C}{6B}\varphi^{(1)2}(\varphi^{(1)} - \varphi_1)(\varphi^{(1)} - \varphi_2), \end{aligned} \quad (33)$$

where

$$\varphi_1 = -\frac{1}{C}(A - \sqrt{\Delta}), \quad \varphi_2 = -\frac{1}{C}(A + \sqrt{\Delta}), \quad (34)$$

$$\text{and } \Delta = A^2 + 6U_0C.$$

We are looking for stationary soliton solutions, whose profile vanishes in the unperturbed region. For this reason, we have used $\varphi \rightarrow 0$ and $d\varphi/d\chi \rightarrow 0$ at $|\chi| \rightarrow \infty$. After integrating Eq. (33), we then obtain positive and negative soliton solutions,²⁶ respectively,

$$\varphi_+^{(1)} = \frac{(6U_0/C)}{\varphi_1 \sinh^2\left(\frac{1}{2}\sqrt{U_0/B\chi}\right) - \varphi_2 \cosh^2\left(\frac{1}{2}\sqrt{U_0/B\chi}\right)}, \quad (35)$$

$$\varphi_-^{(1)} = \frac{(6U_0/C)}{\varphi_2 \sinh^2\left(\frac{1}{2}\sqrt{U_0/B\chi}\right) - \varphi_1 \cosh^2\left(\frac{1}{2}\sqrt{U_0/B\chi}\right)}. \quad (36)$$

To examine the soliton propagation in our e - p - i plasma, A must be in the order of ε and Δ should be positive. The range of the propagation speed (λ), which gives small $A \sim \varepsilon$ and positive Δ , is depicted in Figs. 3(a), 3(b), 4(a), and 4(b) for different values of δ and σ_i . It is seen that for small $\delta=0.1$, the range of λ is narrow for both low and high σ_i . Increasing δ to 0.5, one encounters only supersonic solitons. For $\delta=0.9$, neither subsonic nor supersonic solitons can propagate.

In Fig. 5, we have numerically analyzed the Sagdeev potential (32) and investigated how the propagation speed (λ), the electron-to-ion temperature ratio (σ_i), and the positron-to-electron density ratio (δ) change the profile of the potential well. It turns out that an increase of the propagation speed λ (the electron-to-ion temperature ratio σ_i and positron-to-electron density ratio δ) leads to an increase (a decrease) of the potential depth, but a decrease (increase) of the potential amplitude. Figure 6 shows the dependence of

FIG. 3. (a) The nonlinear coefficient A , (b) the determinant Δ , and (c) the nonlinear coefficient C are depicted against the propagation speed λ for $\sigma_i = 1/3$ (solid curve) and $\sigma_i = 1/300$ (dashed curve), where $\delta = 0.1$.

the compressive and rarefactive solitary pulse on the propagation speed (λ), the electron-to-ion temperature ratio (σ_i), and the positron-to-electron density ratio (δ). It is obvious that a faster positive soliton has a lower amplitude and narrower width. Increasing δ leads to an increase of both the soliton amplitude and the width. However, the amplitude (width) increases (decreases) with increasing σ_i .

B. Double-layer solution

For the double-layer solution, the Sagdeev potential $V(\varphi^{(1)})$ should be negative between $\varphi^{(1)} = 0$ and φ_m , where φ_m is some double root of the potential. Therefore, for the formation of double layers, $V(\varphi^{(1)})$ must satisfy the following conditions:

FIG. 4. (a) The nonlinear coefficient A , (b) the determinant Δ , and (c) the nonlinear coefficient C are depicted against the propagation speed λ for $\sigma_i = 1/3$ (solid curve) and $\sigma_i = 1/300$ (dashed curve), where $\delta = 0.9$.

$$V(\varphi^{(1)}) = 0 \quad \text{at } \varphi^{(1)} = 0 \text{ and } \varphi^{(1)} = \varphi_m, \quad (37)$$

$$V'(\varphi^{(1)}) = 0 \quad \text{at } \varphi^{(1)} = 0 \text{ and } \varphi^{(1)} = \varphi_m, \quad (38)$$

$$V''(\varphi^{(1)}) < 0 \quad \text{at } \varphi^{(1)} = 0 \text{ and } \varphi^{(1)} = \varphi_m. \quad (39)$$

Applying the boundary condition (37) and (38) to Eq. (31), we obtain the conditions for the existence of double layers,

$$U_0 = -\frac{A^2}{6C} \quad \text{and} \quad \varphi_m = -\frac{A}{C}. \quad (40)$$

Notice that these conditions are tantamount to $\Delta = 0$ [cf. Eq. (34) above]. See also that the double-layer propagation speed (correction) U_0 is exactly prescribed by Eq. (40). Substituting U_0 and A from Eq. (40) into Eq. (31), we have

FIG. 5. The Sagdeev potential $V(\varphi^{(1)})$ is depicted against the electrostatic potential $\varphi^{(1)}$ for (a) $\sigma_i=1/3$, $\delta=0.1$, $\lambda=2.35$ (solid curve), $\lambda=2.4$ (dashed curve), (b) $\sigma_i=1/3$, $\lambda=2.35$, $\delta=0.1$ (solid curve), $\delta=0.11$ (dashed curve), (c) $\delta=0.5$, $\lambda=3.5$, $\sigma_i=1/3$ (solid curve), and $\sigma_i=1/4$ (dashed curve).

$$V(\varphi^{(1)}) = \frac{C\varphi^{(1)2}}{12B}(\varphi_m - \varphi^{(1)})^2. \quad (41)$$

Here, $C < 0$ has to be fulfilled in order for $V < 0$, thus ensuring reality of the left-hand side of Eq. (31). Notice that, from Eq. (40), the double root here has the same sign as A . However, negative values of both A and C are partially excluded, as we see in Figs. 2(a), 2(b), 7(a), and 7(c). Thus, double layers will only exist for $A > 0$, and should correspond to positive potential excitations. This is indeed confirmed below.

The solution of Eq. (31), with Eq. (41), is then

FIG. 6. The potential $\varphi^{(1)}$ is depicted against χ for (a) $\sigma_i=1/3$, $\delta=0.1$, $\lambda=2.35$ (solid curve), $\lambda=2.4$ (dashed curve), (b) $\sigma_i=1/3$, $\lambda=2.4$, $\delta=0.1$ (solid curve), $\delta=0.12$ (dashed curve), (c) $\delta=0.5$, $\lambda=3$, $\sigma_i=1/3$ (solid curve), and $\sigma_i=1/4$ (dashed curve).

$$\varphi^{(1)} = \frac{\varphi_m}{2} \left\{ 1 \pm \tanh \left[\varphi_m \left(-\frac{C}{24B} \right)^{1/2} (\xi - U_0\tau) \right] \right\}. \quad (42)$$

The DL solution (42) corresponds to a localized transition among two finite potential regions. Combining Eq. (20) with Eq. (42), we obtain

$$\varphi^{(1)} = \frac{\varphi_m}{2} \left(1 \pm \tanh \left\{ \varphi_m \left(-\frac{C}{24B} \right)^{1/2} \left[\xi - \left(U_0 + \frac{\lambda}{2} \eta^2 \right) \tau \right] \right\} \right). \quad (43)$$

The width of the double layer is

FIG. 7. (a) The nonlinear coefficient A , (b) the determinant Δ , and (c) the nonlinear coefficient C are depicted against the propagation speed λ for $\sigma_i = 1/3$ (solid curve) and $\sigma_i = 1/4$ (dashed curve), where $\delta = 0.5$.

$$W_{DL} = \frac{\sqrt{-24B/C}}{|A/C|}. \quad (44)$$

As we saw above, B is always positive, and the existence of a double layer requires $C < 0$. It is also noted from Eq. (40) that the nature of the double layer depends on the sign of A , i.e., for $A > 0$ a positive double layer exists [viz. $\varphi_m > 0$; see Eq. (40)], whereas for $A < 0$ we would have a negative double layer ($\varphi_m < 0$). The numerical analysis in Figs. 2(a) and 2(c), however, shows that for high λ values, the dominant situation corresponds to $C > 0$ and $A \sim \varepsilon$, so the double layers cannot exist. However, for small λ , double layers may occur, since $C < 0$ and $A \sim \varepsilon$. Furthermore, it is seen that for small λ , the nonlinear coefficient A has a positive value and therefore only positive double layers can exist.

Equation (41) describes the double-layer Sagdeev potential, which has a well-known profile.²⁷ This profile may

FIG. 8. The pseudopotential $V(\varphi^{(1)})$ is depicted against the electrostatic potential $\varphi^{(1)}$ for (a) $\sigma_i = 1/3$, $\delta = 0.5$, $\lambda = 0.1$ (solid curve), $\lambda = 0.2$ (dashed curve), (b) $\sigma_i = 1/3$, $\lambda = 0.2$, $\delta = 0.3$ (solid curve), $\delta = 0.4$ (dashed curve), (c) $\delta = 0.5$, $\lambda = 0.2$, $\sigma_i = 1/3$ (solid curve), and $\sigma_i = 1/300$ (dashed curve).

change due to a variety of physical parameters. We have numerically investigated the effects of the propagation speed (λ), the electron-to-ion temperature ratio (σ_i), and the positron-to-electron density ratio (δ) on the behavior of the Sagdeev pseudopotential (41); see Fig. 8. We recall that the position of the double root determines the amplitude of the excitation, while the potential depth at the minimum here determines the slope (and width) of the excitation; cf. Figs. 8 and 9. It is found [Figs. 8(a) and 9(a)] that increasing the velocity λ leads to a slightly smaller (shorter, in amplitude) excitation. A smaller DL amplitude is also witnessed by increasing the positron concentration, i.e., for higher δ values [see Figs. 8(b) and 9(b)]. Finally, an increase in the ionic temperature (i.e., smaller values of σ_i) shrinks the DL amplitude; in the limit $\sigma_i \rightarrow 0$ (fixed ion limit), no DL excitations

FIG. 9. The double-layer profile $\phi^{(1)}$ is depicted against χ for (a) $\sigma_i=1/3$, $\delta=0.5$, $\lambda=0.1$ (solid curve), and $\lambda=0.2$ (dashed curve), (b) $\sigma_i=1/3$, $\lambda=0.2$, $\delta=0.3$ (solid curve), $\delta=0.4$ (dashed curve), (c) $\delta=0.5$, $\lambda=0.2$, $\sigma_i=1/3$ (solid curve), and $\sigma_i=1/300$ (dashed curve).

are expected to occur. The dependence of double-layer characteristics on the propagation speed (λ), the electron-to-ion temperature ratio (σ_i), and the positron-to-electron density ratio (δ) is displayed in Fig. 9. It is seen that the amplitude and the width of the double layers decrease at high propagation speed. Increasing δ and σ_i would lead to a decrease of the amplitude, but the width would increase slightly. Figure 10 represents the electrostatic potential, as expressed by Eqs. (35), (36), and (42), depicted in three-dimensional plots versus the radial coordinate r and the polar angle θ . It is clear that the shape of both positive and negative soliton pulses as well as shock waves are slightly distorted with polar angle.

FIG. 10. (Color online) The solitary pulse solutions [represented by Eqs. (35) and (36)] and the double-layer profile [represented by Eq. (42)] are depicted against the radial and angular coordinates, r and θ , respectively: (a) compressive pulse, (b) rarefactive pulse (for $\lambda=2.35$, $\sigma_i=1/3$, $\delta=0.1$, $U_0=0.1$, and $t=1$, in both plots), and (c) double-layer profile for $\lambda=0.5$, $\sigma_i=1/3$, $\delta=0.5$, $U_0=0.1$, and $t=1$.

VI. SUMMARY

In this paper, we have studied the nonlinear propagation of electron-acoustic waves in an electron-positron-ion plasma. A background of thermalized ions was considered as a hypothesis that occurs in accretion disks of AGN. By adopting a cylindrical geometry, we have derived a cylindrical Kadomtsev-Petviashvili equation, which yields exact soliton solutions in the form of positive or negative potential pulse excitations.

Furthermore, we have also considered situations in which the strength of the nonlinearity is weak, so that higher-order nonlinearity may play an important role. We have derived an extended cylindrical Kadomtsev-Petviashvili (ECKP) equation, modeling the potential evolution in this case. Exact solutions of the ECKP equation were derived, in the form of localized pulses (which were compared to the CKP pulses obtained previously) and double layers, i.e., propagating localized transition forms (double layers) among regions of constant potential. A set of exact expressions was presented for these localized potential perturbations, and their characteristics were investigated.

The nonlinear structures derived in this study owe their existence to a balance among the dispersion and nonlinearity, and were obtained in a 2D cylindrical geometry (in radial and angular polar coordinates), thus representing structures propagating in the radial direction while being slightly distorted and accelerated due to angular, i.e., (weak) nongyro-tropic effects. Anticipating a more general theoretical scope, it may be pointed out that, from a qualitative point of view, the various scaling assumptions employed here amount to the form $\varphi^{(1)} = \varphi^{(1)}(R)$ for the localized excitation (solitons, double layers), where R is the moving frame defined by

$$R = \varepsilon^{1/2} \left[r - \left(\lambda + \varepsilon U_0 + \frac{\lambda}{2} \theta^2 \right) t \right]. \quad (45)$$

Here, λ is the (main radial) propagation speed, corrected by U_0 , while an angular dependence of the velocity is provided by the last term. The solitary potential excitations thus suffer a velocity modification due to the deviation (here assumed weak) from the radial direction. Thus, these results may be interesting, since they refer to the existence of localized excitations in cylindrical geometries, so their generic characteristics may be traced in different nonlinear dispersive systems.

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